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General conditions for the subadditivity and superadditivity of relations

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Abstract. After some preparations, we give general conditions for a relation on one groupoid to another to be subadditive or superadditive.

1. Introduction

A relation F on one groupoid X to another Y is called subadditive if

$$F(x_1 + x_2) \subset F(x_1) + F(x_2)$$

for all $x_1, x_2 \in X$. If the inclusion is reversed, then F is called superadditive.

The importance of these definitions lies mainly in the fact that Banach's closed graph and open mapping theorems can be, most naturally, formulated in terms of superadditive relations. The corresponding treatments of the Banach–Steinhaus, Hahn–Banach and Hyers–Ulam theorems need subadditive relations. The 'References' of our former paper [**2**] show that subadditive and superadditive relations have been intensively studied by several authors. In [**2**], we have proved the following theorem.

Theorem 1.1. *Let F be an odd relation on one group X to another Y. Suppose that there exists a subset P of Y, such that $Y = -P \cup P$, and*

$$F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$$

for all $x_1, x_2 \in X$. Then F is subadditive.

Now, having in mind the anti-additive functions defined by

$$\varphi(x) = -x \quad \text{for all} \quad x \in X \qquad \text{and} \qquad \psi(y) = -y \quad \text{for all} \quad y \in Y,$$

and the dualization of groupoids, we shall prove the following generalization of the above theorem.

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Theorem 1.2. *Let F be a relation on one groupoid X to another Y. Suppose that there exists a relation G on one groupoid Z to another W, a subadditive relation φ on X to Z, an additive function ψ on W to Y, and moreover, subsets P of Y and Q of W, such that:*

(1) $\psi^{-1} \circ F = G \circ \varphi$;
(2) $R_F \setminus P \subset D_{\psi^{-1}}$ and $\psi^{-1}\big[R_F \setminus P\big] \subset Q$;
(3) $F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$ for all $x_1, x_2 \in X$;
(4) $G(z_1 + z_2) \cap Q \subset G(z_1) + G(z_2)$ for all $z_1, z_2 \in Z$.

Then F is subadditive.

Analogously to Theorem 1.2, we shall also prove the following result.

Theorem 1.3. *Let F be a relation on a groupoid X to a group Y such that $R_F + R_F \subset R_F$. Suppose that there exists a relation G on a groupoid Z to a group W, a superadditive relation φ on X to Z, an odd additive function ψ on W onto Y, and moreover, subsets P of Y and Q of W such that:*

(1) $\psi^{-1} \circ F = G \circ \varphi$;
(2) $\psi^{-1}\big[R_F \setminus P\big] \subset Q$;
(3) $\big(F(x_1) + F(x_2)\big) \cap P \subset F(x_1 + x_2)$ for all $x_1, x_2 \in X$;
(4) $\big(G(z_1) + G(z_2)\big) \cap Q \subset G(z_1 + z_2)$ for all $z_1, z_2 \in Z$.

Then F is superadditive.

Hence, we can easily derive the following counterpart of Theorem 1.1.

Theorem 1.4. *Let F be an odd relation on one group X to another Y, such that $R_F + R_F \subset R_F$. Suppose that there exists a subset P of Y such that $Y = -P \cup P$ and*

$$\big(F(x_1) + F(x_2)\big) \cap P \subset F(x_1 + x_2)$$

for all $x_1, x_2 \in X$. Then F is superadditive.

For the reader's convenience, the necessary prerequisites concerning relations and groupoids will be briefly laid out in the following preparatory sections.

2. A few basic facts on relations

A subset F of the product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. Thus, $\Delta_X = \{(x,x) : x \in X\}$ is a relation on X.

If F is a relation on X to Y, then for any $x \in X$ the set $F(x) = \{y \in Y : (x,y) \in F\}$ is called the image of x under F. And the set $D_F = \{x \in X : F(x) \neq \emptyset\}$ is called the domain of F.

In particular, a relation F on X to Y is called a function if for each $x \in D_F$ there exists $y \in Y$, such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we may usually write $F(x) = y$ in place of $F(x) = \{y\}$.

More generally, if F is a relation on X to Y, then for any $A \subset X$ the set $F[A] = \bigcup_{x \in A} F(x)$ is called the image of A under F. And the set $R_F = F[D_F]$ is called the range of F.

If F is a relation such that $D_F = X$ and $R_F \subset Y$, then we say that F is a relation of X into Y. While, if F is relation such that $D_F \subset X$ and $R_F = Y$, then we say that F is a relation on X onto Y.
If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since $F = \bigcup_{x\in X}\{x\}\times F(x)$. Therefore, the inverse F^{-1} of F can be defined such that $F^{-1}(y) = \big\{x \in X : \ y \in F(x)\big\}$ for all $y \in Y$.

Moreover, if F is a relation on X to Y and G is a relation on Y to Z, then the composition $G \circ F$ of G and F can be defined such that $(G \circ F)(x) = G\big[F(x)\big]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subset X$.

Concerning relations, in the sequel, we shall also need the following

Theorem 2.1. *If F is a relation on X to Y, then*

$$\Delta_{R_F} \subset F \circ F^{-1}.$$

Proof. If $y \in R_F$, then there exists $x \in X$ such that $y \in F(x)$, and thus, $x \in F^{-1}(y)$. Hence, we can see that

$$\Delta_{R_F}(y) = \{y\} \subset F(x) \subset F\big[F^{-1}(y)\big] = \big(F \circ F^{-1}\big)(y).$$

Now, since $\Delta_{R_F}(y) = \emptyset$ for all $y \in Y \setminus R_F$, it is clear that we actually have $\Delta_{R_F}(y) \subset \big(F \circ F^{-1}\big)(y)$ for all $y \in Y$. Therefore, the required inclusion is also true. □

Now, as an immediate consequence of the above theorem, we can also state

Corollary 2.1. *If F is a relation on X to Y, then*

$$A \cap R_F \subset F\big[F^{-1}[A]\big]$$

for all $A \subset Y$.

Hence, we can easily derive the following two corollaries

Corollary 2.2. *If F is a relation on X to Y and $A \subset Y$, then the following assertions are equivalent:*

(1) $A \subset R_F$;
(2) $A \subset F\big[F^{-1}[A]\big]$.

Remark 2.1. Thus, in particular, for any $y \in Y$ we have $y \in F\big[F^{-1}(y)\big]$ if and only if $y \in R_F$.

Corollary 2.3. *If F is a relation on X to Y, then*

$$R_F = R_{F\circ F^{-1}} = D_{F\circ F^{-1}}.$$

Proof. To check this, note that

$$R_F = Y \cap R_F \subset F\big[F^{-1}[Y]\big] = F[D_F] = R_F,$$

and thus,

$$R_F = \big(F \circ F^{-1}\big)[Y] = R_{F\circ F^{-1}} = D_{(F\circ F^{-1})^{-1}} = D_{F\circ F^{-1}}.$$

□

3. Some useful characterizations of functions

In the sequel, in addition to Theorem 2.1, we shall also need the following

THEOREM 3.1. *If F is a relation on X to Y, then the following assertions are equivalent:*

(1) *F is a function;*
(2) *$F \circ F^{-1} \subset \Delta_Y$;*
(3) *$\Delta_{R_F} = F \circ F^{-1}$.*

PROOF. If $y \in Y$ and $z \in \big(F \circ F^{-1}\big)(y)$, then $z \in F\big[F^{-1}(y)\big]$. Therefore, there exists $x \in F^{-1}(y)$, such that $z \in F(x)$. This shows that $y, z \in F(x)$. Hence, if (1) holds, we can infer that $z = y$. Therefore,

$$\big(F \circ F^{-1}\big)(y) \subset \{y\} = \Delta_Y(y),$$

and thus (2) also holds.

To prove the converse implication, note that if (1) does not hold, then there exist $x \in X$ and $y, z \in F(x)$ such that $y \neq z$. Hence, we can infer that $x \in F^{-1}(y)$, and thus

$$\{y, z\} \subset F(x) \subset F\big[F^{-1}(y)\big] = \big(F \circ F^{-1}\big)(y).$$

Therefore, since $\Delta_Y(y) = \{y\}$, the assertion (2) does not also holds.

Finally, to complete the proof, note that if (2) holds, then by Corollary 2.3, we also have $F \circ F^{-1} \subset \Delta_{R_F}$. Hence, by Theorem 2.1, it is clear that (3) also holds. Moreover, (3) trivially implies (2). □

Now, as an immediate consequence of the above theorem, we can also state

COROLLARY 3.1. *If F is a relation on X to Y, then the following assertions are equivalent:*

(1) *F is a function;*
(2) *$F\big[F^{-1}[A]\big] \subset A$ for all $A \subset Y$;*
(3) *$A \cap R_F = F\big[F^{-1}[A]\big]$ for all $A \subset Y$.*

REMARK 3.1. Note that already the $A = \{y\}$, where y runs through R_F, particular cases of (2) and (3) imply (1).

In addition to the above corollary, it is also worth mentioning the following two theorems.

THEOREM 3.2. *If F is a relation on X to Y, then the following assertions are equivalent:*

(1) *F is a function;*
(2) *$F^{-1}[A] \cap F^{-1}[B] \subset F^{-1}[A \cap B]$ for all $A, B \subset X$;*
(3) *$F^{-1}[A \cap B] = F^{-1}[A] \cap F^{-1}[B]$ for all $A, B \subset X$.*

THEOREM 3.3. *If F is a relation on X to Y, then the following assertions are equivalent:*

(1) *F is a function;*
(2) *$F[A \setminus B] \subset F^{-1}[A] \setminus F^{-1}[B]$ for all $A, B \subset X$;*

(3) $F^{-1}[A \setminus B] = F^{-1}[A] \setminus F^{-1}[B]$ *for all* $A, B \subset X$.

Remark 3.2. Note that if (1) does not hold, then there exists $x \in X$ and $y, z \in F(x)$, such that $y \neq z$. Therefore, $x \in F^{-1}(y) \cap F^{-1}(z)$, but $x \notin \emptyset = F^{-1}[\emptyset] = F^{-1}[\{y\} \cap \{z\}]$. Moreover, $x \in F^{-1}(y) \subset F^{-1}[Y \setminus \{z\}]$, but $x \notin F^{-1}[Y] \setminus F^{-1}(z)$. Thus, already some very particular cases of assertions (2) in the above two theorems imply assertion (1).

4. A few basic facts on subadditive and superadditive relations

If X is a nonvoid set and $+$ is a function of $X^2 = X \times X$ into X, then the ordered pair $X(+) = (X, +)$ is called a groupoid. In this case, we may also write $x + y = +(x, y)$ for all $x, y \in X$. Moreover, if X is a groupoid, then we may also naturally write $A + B = \big\{x + y : \ x \in A, \ y \in B\big\}$ for all $A, B \subset X$. Thus, the family $\mathcal{P}(X)$ of all subsets of X is also a groupoid.

Note that if X is, in particular, a group, then $\mathcal{P}(X)$ is, in general, only a semigroup with zero element $\{0\}$. However, we can still naturally use the notations $-A = \big\{-x : x \in A\big\}$ and $A - B = A + (-B)$.

Definition 4.1. A relation F on one groupoid X to another Y is called subadditive if

$$F(x_1 + x_2) \subset F(x_1) + F(x_2)$$

for all $x_1, x_2 \in X$. If the inclusion is reversed, then F is called superadditive.

Remark 4.1. Now, the relation F may be naturally called additive if it is both subadditive and superadditive.

Definition 4.2. A relation F on one group X to a group Y is called odd if $F(-x) = -F(x)$ for all $x \in X$.

Remark 4.2. Quite similarly, a relation F on a group X to a set Y is called even if $F(-x) = F(x)$ for all $x \in X$.

The importance of odd relation is apparent from the following

Theorem 4.1. *If F is an odd superadditive relation of one group X into another Y, then F is additive.*

Proof. If $x_1, x_2 \in X$, then by taking $y \in F(x_1)$, we can easily see that

$$F(x_1 + x_2) = y - y + F(x_1 + x_2) \in F(x_1) - F(x_1) + F(x_1 + x_2) =$$

$$= F(x_1) + F(-x_1) + F(x_1 + x_2) \subset F(x_1) + F(x_2).$$

Therefore, the equality $F(x_1 + x_2) = F(x_1) + F(x_2)$ is also true. □

Remark 4.3. In this respect, it is also worth to remark that an additive function f on X to Y is odd if and only if its domain D_f is symmetric in the sense that $-D_f = D_f$.

Concerning the superadditivity of inverse relations, we can easily establish the following theorem

THEOREM 4.2. *If F is a superadditive relation on one groupoid X to another Y, then F^{-1} is a superadditive relation on Y to X.*

REMARK 4.4. By defining $F(x) = \{0, x\}$ for all $x \in \mathbb{R}$, one can easily see that the counterparts of Theorems 4.1 and 4.2 are not true for subadditive relations.

However, analogously to Theorem 4.2, we also have the following theorem

THEOREM 4.3. *If F is an odd relation on one group X to another Y, then F^{-1} is an odd relation on Y to X.*

REMARK 4.5. In this respect, it is also worth to note that if F is a relation on a set X to a group Y, then F^{-1} is even if and only if F is symmetric-valued in the sense that $F(x)$ is symmetric for all $x \in X$.

Now, as an immediate consequence of Theorems 4.3, 4.2 and 4.1, we can also state

THEOREM 4.4. *If F is an odd superadditive relation on one group X onto another Y, then F^{-1} is an odd additive relation of Y into X.*

5. Some further results on subadditive and superadditive relations

THEOREM 5.1. *If F is a relation on one groupoid X to another Y and $P \subset Y$, such that $F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$ for all $x_1, x_2 \in X$, then*

$$F[A_1 + A_2] \cap P \subset F[A_1] + F[A_2]$$

for all $A_1, A_2 \subset X$.

PROOF. If $A_1, A_2 \subset X$ and $y \in F[A_1 + A_2] \cap P$, then $y \in F[A_1 + A_2]$ and $y \in P$. Therefore, there exist $x_1 \in A_1$ and $x_2 \in A_2$, such that $y \in F(x_1 + x_2)$. Hence, it is clear that $y \in F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2) \subset F[A_1] + F[A_2]$. Therefore, $F[A_1 + A_2] \cap P \subset F[A_1] + F[A_2]$. □

The particular case of the above theorem for $P = Y$ gives

COROLLARY 5.1. *If F is a subadditive relation on one groupoid X to another Y, then*

$$F[A_1 + A_2] \subset F[A_1] + F[A_2]$$

for all $A_1, A_2 \subset X$.

Analogously to Theorem 5.1, we can also prove the following two theorems

THEOREM 5.2. *If F is a relation on one groupoid X to another Y and $P \subset Y$, such that $\big(F(x_1) + F(x_2)\big) \cap P \subset F(x_1 + x_2)$ for all $x_1, x_2 \in X$, then*

$$\big(F[A_1] + F[A_2]\big) \cap P \subset F[A_1 + A_2]$$

for all $A_1, A_2 \subset X$.

THEOREM 5.3. *If F is a relation on one groupoid X to another Y and $P_1, P_2 \subset Y$, such that $F(x_1) \cap P_1 + F(x_2) \cap P_2 \subset F(x_1 + x_2)$ for all $x_1, x_2 \in X$, then*

$$F[A_1] \cap P_1 + F[A_2] \cap P_2 \subset F[A_1 + A_2]$$

for all $A_1, A_2 \subset X$.

PROOF. If $A_1, A_2 \subset X$ and $y \in F[A_1] \cap P_1 + F[A_2] \cap P_2$, then for each $i \in \{1, 2\}$ there exists $y_i \in F[A_i] \cap P_i$, such that $y = y_1 + y_2$. Hence, it follows that $y_i \in F[A_i]$ and $y_i \in P_i$. Therefore, there exists $x_i \in A_i$, such that $y_i \in F(x_i)$. Hence, it is clear that

$$y = y_1 + y_2 \in F(x_1) \cap P_1 + F(x_2) \cap P_2 \subset F(x_1 + x_2) \subset F[A_1 + A_2].$$

Therefore, $F[A_1] \cap P_1 + F[A_2] \cap P_2 \subset F[A_1 + A_2]$. □

The particular cases of the above two theorems for $P = Y$ and $P_1 = P_2 = Y$ give

COROLLARY 5.2. *If F is a superadditive relation on one groupoid X to another Y, then*

$$F[A_1] + F[A_2] \subset F[A_1 + A_2]$$

for all $A_1, A_2 \subset X$.

REMARK 5.1. Moreover, we can easily see that if F is an odd relation on one group X to another Y, then $F[-A] = -F[A]$ for all $A \subset X$.

Now as an immediate consequence of Theorem 4.4 and Corollaries 5.1 and 5.2, we can also state the following

THEOREM 5.4. *If F is an odd superadditive relation on one group X onto another Y, then*

$$F^{-1}[B_1 + B_2] = F^{-1}[B_1] + F^{-1}[B_2]$$

for all $B_1, B_2 \subset Y$.

Finally, we note that in addition to Definition 4.1, it is also worth introducing the following

DEFINITION 5.1. A relation F on one groupoid X to another Y is called anti-subadditive if

$$F(x_1 + x_2) \subset F(x_2) + F(x_1)$$

for all $x_1, x_2 \in X$. If the inclusion is reversed, then F is called antisuperadditive.

REMARK 5.2. Namely, if X is a group and $\varphi(x) = -x$ for all $x \in X$, then φ is an anti-additive function of X onto itself.

Fortunately, the study of anti-subadditive and anti-superadditive relations can be traced back to that of the subadditive and superadditive ones by means of the following definition

DEFINITION 5.2. If $X(+)$ is a groupoid and

$$x_1 \oplus x_2 = x_2 + x_1$$

for all $x_1, x_2 \in X$, then the groupoid $X(\oplus)$ will be called the dual of $X(+)$.

REMARK 5.3. Note that if $X(+)$ has a zero element 0, then 0 is also a zero element of $X(\oplus)$. Moreover, if an element x of $X(+)$ has an inverse element $-x$, then $-x$ is also the inverse element of x in $X(\oplus)$.

6. Some general conditions for the subadditivity of relations

Theorem 6.1. *Let F be a relation on one groupoid X to another Y. Suppose that there exists a relation G on one groupoid Z to another W, a subadditive (anti-subadditive) relation φ on X to Z, an additive (anti-additive) function ψ on W to Y, and moreover, subsets P of Y and Q of W such that:*

(1) $\psi^{-1} \circ F = G \circ \varphi$;
(2) $R_F \setminus P \subset D_{\psi^{-1}}$ *and* $\psi^{-1}\big[R_F \setminus P\big] \subset Q$;
(3) $F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$ *for all* $x_1, x_2 \in X$;
(4) $G(z_1 + z_2) \cap Q \subset G(z_1) + G(z_2)$ *for all* $z_1, z_2 \in Z$.

Then F is subadditive.

Proof. Suppose that $x_1, x_2 \in X$ and $y \in F(x_1 + x_2)$. If $y \in P$, then $y \in F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$. Therefore, we may assume that $y \notin P$. In this case, because of $F(x_1 + x_2) \subset R_F$, it is clear that

$$y \in F(x_1 + x_2) \cap (Y \setminus P) = F(x_1 + x_2) \cap (R_F \setminus P) \subset R_F \setminus P \subset D_{\psi^{-1}} = R_\psi.$$

Moreover, if φ is subadditive and ψ is additive, then by using Theorems 3.2, 5.1, and 4.2 and Corollary 5.2, we can see that

$$\psi^{-1}(y) \subset \psi^{-1}\big[F(x_1 + x_2) \cap (R_F \setminus P)\big] = \psi^{-1}\big[F(x_1 + x_2)\big] \cap \psi^{-1}\big[R_F \setminus P\big] \subset$$
$$G\big[\varphi(x_1 + x_2)\big] \cap Q \subset G\big[\varphi(x_1) + \varphi(x_2)\big] \cap Q \subset G\big[\varphi(x_1)\big] + G\big[\varphi(x_2)\big] =$$
$$\psi^{-1}\big[F(x_1)\big] + \psi^{-1}\big[F(x_2)\big] \subset \psi^{-1}\big[F(x_1) + F(x_2)\big].$$

Hence, by Remark 2.1 and Corollary 3.1, it is clear that

$$y \in \psi\big[\psi^{-1}(y)\big] \subset \psi\left[\psi^{-1}\big[F(x_1) + F(x_2)\big]\right] \subset F(x_1) + F(x_2).$$

Therefore, $F(x_1 + x_2) \subset F(x_1) + F(x_2)$, and thus, F is subadditive.

To prove the anti-additive case of the theorem, note that if φ is an anti-subadditive relation on $X(+)$ to $Z(+)$, then φ is a subadditive relation on $X(+)$ to $Z(\oplus)$. And, if ψ is an anti-additivity function on $W(+)$ to $Y(+)$, then ψ is an additive function on $W(\oplus)$ to $Y(+)$. Moreover, because of (4), we have

$$G(z_1 \oplus z_2) \cap Q = G(z_2 + z_1) \cap Q \subset G(z_2) + G(z_1) = G(z_1) \oplus G(z_2)$$

for all $z_1, z_2 \in Z$. Therefore, the additive case of the theorem can be applied to get the subadditivity of F. □

Remark 6.1. Because of $\psi^{-1} \circ F = G \circ \varphi$, we also have

$$\psi^{-1}\big[R_F\big] = \psi^{-1}\big[F[X]\big] = G\big[\varphi[X]\big] = G\big[R_\varphi\big].$$

Hence, by noticing that

$$\psi^{-1}\big[R_F\big] \subset \psi^{-1}\big[Y\big] = D_\psi \qquad \text{and} \qquad G\big[R_\varphi\big] \subset G\big[Z\big] = R_G,$$

we can infer that

$$\psi^{-1}\big[R_F \setminus P\big] \subset \psi^{-1}\big[R_F\big] \subset D_\psi \cap R_G.$$

Therefore, if in particular $D_\psi \cap R_G \subset Q$, then the inclusion $\psi^{-1}\big[R_F \setminus P\big] \subset Q$ holds.

From the particular case of Theorem 6.1 for $X = Z$, $Y = W$, $P = Q$ and $F = G$ we can immediately get

Corollary 6.1. *Let F be a relation on one groupoid X to another Y. Suppose that there exist a subadditive (anti-subadditive) relation φ on X to X, an additive (anti-additive) function ψ on Y to Y, and P is a subset of Y such that:*

(1) $\psi^{-1} \circ F = F \circ \varphi$;
(2) $R_F \setminus P \subset D_{\psi^{-1}}$ *and* $\psi^{-1}\big[R_F \setminus P\big] \subset P$;
(3) $F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$ *for all* $x_1, x_2 \in X$.

Then F is subadditive.

Now, as an useful consequence Corollary 6.1, we can also state

Corollary 6.2. *Let F be a symmetric-valued relation on a groupoid X to a group Y. Suppose that either X or Y is commutative, and there exists a subset P of Y, such that $Y = -P \cup P$ and*

$$F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$$

for all $x_1, x_2 \in X$. Then F is subadditive.

Proof. Now, $-F(x) = F(x)$ for all $x \in X$. Therefore, by taking

$$\varphi(x) = x \quad \text{for all} \quad x \in X \qquad \text{and} \qquad \psi(y) = -y \quad \text{for all} \quad y \in Y,$$

we evidently have $\psi^{-1} \circ F = F \circ \varphi$. On the other hand, now $D_{\psi^{-1}} = Y$. Therefore, $R_F \setminus P \subset D_{\psi^{-1}}$ trivially holds. Moreover, since $Y = -P \cup P$, it is clear that $\psi^{-1}\big[Y \setminus P\big] \subset P$ also holds. Therefore, depending on the commutativity of Y or X the additive or the anti-additive case of Corollary 6.1 can be applied. □

Remark 6.2. Recall that, by Remark 4.5, the relation F is symmetric-valued if and only if its inverse F^{-1} is even.

From the anti-additive case of Corollary 6.1, we can also easily get the following statement

Corollary 6.3. *Let F be an odd relation on one group X to another Y. Suppose that there exists a subset P of Y, such that $Y = -P \cup P$ and*

$$F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$$

for all $x_1, x_2 \in X$. Then F is subadditive.

Now, by using this corollary, we can also prove the following one

Corollary 6.4. *Let F be an odd relation of one group X into another such that $F^{-1}(0) \subset \{0\}$. Suppose that there exists a subset P of Y, such that $Y = -P \cup \{0\} \cup P$ and*

$$F(x_1 + x_2) \cap P \subset F(x_1) + F(x_2)$$

for all $x_1, x_2 \in X$. Then F is subadditive.

PROOF. If $x_1, x_2 \in X$ and $0 \in F(x_1+x_2)$, then $x_1+x_2 \in F^{-1}(0) \subset \{0\}$. Therefore, $x_1+x_2=0$, and thus, $-x_1=x_2$. Hence, by taking $y \in F(x_1)$, we can already infer that

$$0 = y - y \in F(x_1) - F(x_1) = F(x_1) + F(-x_1) = F(x_1) + F(x_2).$$

Therefore, under the notation $P' = P \cup \{0\}$, we have not only $Y = -P' \cup P'$, but also $F(x_1+x_2) \cap P' \subset F(x_1) + F(x_2)$ for all $x_1, x_2 \in X$. Thus, Corollary 6.3 can be applied. □

REMARK 6.3. Note that if, in particular, $Y = \mathbb{R} =]-\infty, +\infty[$, then we may naturally take $P = \mathbb{R}_+ =]0, +\infty[$.

7. Some general conditions for the superadditivity of relations

Analogously to Theorem 6.1, we can also prove the following statement

THEOREM 7.1. *Let F be a relation on a groupoid X to a group Y, such that $R_F + R_F \subset R_F$. Suppose that there exists a relation G on a groupoid Z to a group W, a superadditive (anti-superadditive) relation φ on X to Z, an odd additive (anti-additive) function ψ on W onto Y, and moreover subsets P of Y and Q of W such that:*

(1) $\psi^{-1} \circ F = G \circ \varphi$;
(2) $\psi^{-1}\big[R_F \setminus P\big] \subset Q$;
(3) $\big(F(x_1) + F(x_2)\big) \cap P \subset F(x_1+x_2)$ *for all* $x_1, x_2 \in X$;
(4) $\big(G(z_1) + G(z_2)\big) \cap Q \subset G(z_1+z_2)$ *for all* $z_1, z_2 \in Z$

Then F is superadditive.

PROOF. Suppose that $x_1, x_2 \in X$ and $y \in F(x_1) + F(x_2)$. If $y \in P$, then $y \in \big(F(x_1) + F(x_2)\big) \cap P \subset F(x_1+x_2)$. Therefore, we may assume that $y \notin P$. In this case, because of $F(x_1) + F(x_2) \subset R_F + R_F \subset R_F$, it is clear that

$$y \in \big(F(x_1) + F(x_2)\big) \cap (Y \setminus P) = \big(F(x_1) + F(x_2)\big) \cap (R_F \setminus P) \subset Y = R_\psi.$$

Moreover, if φ is superadditive and ψ is additive, then by using Theorems 3.2 and 5.4 and 5.2, we can see that

$$\psi^{-1}(y) \subset \psi^{-1}\Big[\big(F(x_1) + F(x_2)\big) \cap (R_F \setminus P)\Big] =$$
$$\psi^{-1}\big[F(x_1) + F(x_2)\big] \cap \psi^{-1}\big[R_F \setminus P\big] \subset \psi^{-1}\big[F(x_1) + F(x_2)\big] \cap Q =$$
$$\Big(\psi^{-1}\big[F(x_1)\big] + \psi^{-1}\big[F(x_2)\big]\Big) \cap Q = \Big(G\big[\varphi(x_1)\big] + G\big[\varphi(x_2)\big]\Big) \cap Q \subset$$
$$G\big[\varphi(x_1) + \varphi(x_2)\big] \subset G\big[\varphi(x_1+x_2)\big] = \psi^{-1}\big[F(x_1+x_2)\big].$$

Hence, by Remark 2.1 and Corollary 3.1, it is clear that

$$y \in \psi\big[\psi^{-1}(y)\big] \subset \psi\left[\psi^{-1}\big[F(x_1+x_2)\big]\right] \subset F(x_1+x_2).$$

Therefore, $F(x_1) + F(x_2) \subset F(x_1+x_2)$, and thus, F is superadditive.

The anti-additive case of the theorem, can again be easily derived from its additive case by using the duals of Z and W. □

From Theorem 7.1 we can immediately get the following corollary.

COROLLARY 7.1. *Let F be a relation on a groupoid X to a group Y, such that $R_F + R_F \subset R_F$. Suppose that there exists a superadditive (anti-superadditive) relation φ on X to X, an odd additive (anti-additive) function ψ on Y onto Y, and a subset P of Y such that:*

(1) $\psi^{-1} \circ F = G \circ \varphi$;
(2) $\psi^{-1}\big[R_F \setminus P\big] \subset P$;
(3) $\big(F(x_1) + F(x_2)\big) \cap P \subset F(x_1 + x_2)$ *for all* $x_1, x_2 \in X$.

Then F is superadditive.

Now, as a useful consequence of Corollary 7.1 we can also state the following

COROLLARY 7.2. *Let F be a symmetric-valued relation on a groupoid X to a group Y such that $R_F + R_F \subset R_F$. Suppose that either X or Y is commutative, and there exists a subset P of Y, such that $Y = -P \cup P$ and*

$$\big(F(x_1) + F(x_2)\big) \cap P \subset F(x_1 + x_2)$$

for all $x_1, x_2 \in X$. Then F is superadditive.

From the anti-additive case of Corollary 7.1, we get the following

COROLLARY 7.3. *Let F be an odd relation on one group X to another Y, such that $R_F + R_F \subset R_F$. Suppose that there exists a subset P of Y, such that $Y = -P \cup P$ and*

$$\big(F(x_1) + F(x_2)\big) \cap P \subset F(x_1 + x_2)$$

for all $x_1, x_2 \in X$. Then F is superadditive.

8. Some further conditions for the superadditivity of relations

Analogously to Theorem 7.1, we obtain the following statement.

THEOREM 8.1. *Let F be a relation on a groupoid X to a group Y. Suppose that there exists a relation G on a groupoid Z to a group W, a superadditive relation φ on X to Z, an odd additive function ψ on W onto Y, and moreover subsets P of Y and Q of W such that:*

(1) $\psi^{-1} \circ F = G \circ \varphi$;
(2) $\psi^{-1}\big[R_F \setminus P\big] \subset Q$;
(3) $F(x_1) \cap P + F(x_2) \subset F(x_1 + x_2)$ *for all* $x_1, x_2 \in X$;
(4) $G(z_1) \cap Q + G(z_2) \subset G(z_1 + z_2)$ *for all* $z_1, z_2 \in Z$.

Then F is superadditive.

PROOF. Suppose that $x_1, x_2 \in X$ and $y \in F(x_1) + F(x_2)$. Then, there exist $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$, such that $y = y_1 + y_2$. If $y_1 \in P$, then $y = y_1 + y_2 \in F(x_1) \cap P + F(x_2) \subset F(x_1 + x_2)$. Therefore, we may assume that $y_1 \notin P$. In this case, because of $F(x_1) \subset R_F$, it is clear that

$$y = y_1 + y_2 \in F(x_1) \cap (Y \setminus P) + F(x_2) = F(x_1) \cap (R_F \setminus P) + F(x_2) \subset Y = R_\psi.$$

Moreover, by using Theorems 5.4, 3.2, and 5.3, we can see that

$$\psi^{-1}(y) \subset \psi^{-1}\big[F(x_1) \cap (R_F \setminus P) + F(x_2)\big] = \psi^{-1}\big[F(x_1) \cap (R_F \setminus P)\big] + \psi^{-1}\big[F(x_2)\big] =$$

$$\psi^{-1}\big[F(x_1)\big] \cap \psi^{-1}\big[R_F \setminus P\big] + \psi^{-1}\big[F(x_2)\big] \subset G\big[\varphi(x_2)\big] \cap Q + G\big[\varphi(x_1)\big] \subset G\big[\varphi(x_1) + \varphi(x_2)\big] \subset G\big[\varphi(x_1 + x_2)\big] = \psi^{-1}\big[F(x_1 + x_2)\big].$$

Hence, by Remark 2.1 and Corollary 3.1, it is clear that

$$y \in \psi\big[\psi^{-1}(y)\big] \subset \psi\,\big[\psi^{-1}\big[F(x_1 + x_2)\big]\big] \subset F(x_1 + x_2).$$

Therefore, $F(x_1) + F(x_2) \subset F(x_1 + x_2)$, and thus F is superadditive. □

From Theorem 8.1 we can immediately get the following corollary.

COROLLARY 8.1. *Let F be a relation on a groupoid X to a group Y. Suppose that there exists a superadditive relation φ on X to X, an odd additive function ψ on Y onto Y, and a subset P of Y such that:*

(1) $\psi^{-1} \circ F = G \circ \varphi$;
(2) $\psi^{-1}\big[R_F \setminus P\big] \subset P$;
(3) $F(x_1) \cap P + F(x_2) \subset F(x_1 + x_2)$ *for all* $x_1, x_2 \in X$.

Then F is superadditive.

Hence, it is clear that in particular we also have

COROLLARY 8.2. *Let F be a symmetric-valued relation on a groupoid X to a commutative group Y. Suppose that there exists a subset P of Y, such that $Y = -P \cup P$ and*

$$F(x_1) \cap P + F(x_2) \subset F(x_1 + x_2)$$

for all $x_1, x_2 \in X$. Then F is superadditive.

Moreover, from Theorem 8.1, we easily get the following statement

THEOREM 8.2. *Let F be a relation on a groupoid X to a group Y. Suppose that there exists a relation G on a groupoid Z to a group W, an anti-superadditive relation φ on X to Z, an odd anti-additive function ψ on W onto Y, and moreover subsets P of Y and Q of W, such that:*

(1) $\psi^{-1} \circ F = G \circ \varphi$;
(2) $\psi^{-1}\big[R_F \setminus P\big] \subset Q$;
(3) $F(x_1) \cap P + F(x_2) \subset F(x_1 + x_2)$ *for all* $x_1, x_2 \in X$;
(4) $G(z_1) + G(z_2) \cap Q \subset G(z_1 + z_2)$ *for all* $z_1, z_2 \in Z$.

Then F is superadditive.

PROOF. In this case, φ is a superadditive relation on $X(+)$ to $Z(\oplus)$ and ψ is a odd additive function on $W(\oplus)$ onto $Y(+)$. Moreover, we have

$$G(z_1) \cap Q \oplus G(z_2) = G(z_2) + G(z_1) \cap Q \subset G(z_2 + z_1) = G(z_1 \oplus z_2)$$

for all $z_1, z_2 \in Z$. Therefore, Theorem 8.1 can be applied to get the superadditivity of F. □

From Theorem 8.2 we can immediately get the following corollary.

COROLLARY 8.3. *Let F be a relation on a groupoid X to a group Y. Suppose that there exists an anti-superadditive relation φ on X to X, an odd anti-additive function ψ on Y onto Y, and moreover subsets P and Q of Y, such that:*

(1) *$\psi^{-1} \circ F = G \circ \varphi$;*
(2) *$\psi^{-1}\big[R_F \setminus P\big] \subset Q$;*
(3) *$(F(x_1) \cap P + F(x_2)) \cup (F(x_1) + F(x_2) \cap Q) \subset F(x_1 + x_2)$ for all $x_1, x_2 \in X$.*
Then F is superadditive.

Hence, it is clear that in particular we also have

Corollary 8.4. *Let F be an odd relation on one group X to another Y. Suppose that there exists a subset P of Y such that $Y = -P \cup P$ and*

$$(F(x_1) \cap P + F(x_2)) \cup (F(x_1) + F(x_2) \cap P) \subset F(x_1 + x_2)$$

for all $x_1, x_2 \in X$. Then F is superadditive.

Remark 8.1. To obtain some further conditions for the superadditivity of F, we can note that in Theorem 8.1, instead of conditions (3) and (4), we may assume that

(3') $F(x_1) + F(x_2) \cap P \subset F(x_1 + x_2)$ for all $x_1, x_2 \in X$;
(4') $G(z_1) + G(z_2) \cap Q \subset G(z_1 + z_2)$ for all $z_1, z_2 \in Z$.

Namely, if φ and ψ are as in Theorem 8.1, then φ is also a superadditive relation on $X(\oplus)$ to $Z(\oplus)$ and ψ is also an odd additive function on $W(\oplus)$ onto $Y(\oplus)$. Moreover, if (3') and (4') hold, then we have

(3") $F(x_1) \cap P \oplus F(x_2) \subset F(x_1 \oplus x_2)$ for all $x_1, x_2 \in X$;
(4") $G(z_1) \cap Q \oplus G(z_2) \subset G(z_1 \oplus z_2)$ for all $z_1, z_2 \in Z$.

Therefore, Theorem 7.1 can be applied with $X(\oplus)$, $Y(\oplus)$, $Z(\oplus)$, and $W(\oplus)$ in place of $X(+)$, $Y(+)$, $Z(+)$, and $W(+)$, respectively.

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