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A Schottky description of a Theorem of Conder-Maclahlan-Vasiljevic-Wilson

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Abstract. In a recently paper of Conder-Maclahlan-Vasiljevic-Wilson [**7**]it has been proved that for every positive integer $g \geqslant 2$ there exists a closed non-orientable surface of algebraic genus g with at least $4(g+1)$ automorphisms if g is even, or at least $8(g-1)$ automorphisms if g is odd. The main purpose of this note is to provide explicitly such kind of situations in terms of Schottky groups. We also provide a construction of closed non-orientable surfaces of algebraic genus g, for infinite many values of integers $g \geqslant 2$, so that they admit a group of automorphisms of order $12(g-1)$ which can be reflected by Schottky groups.

1. Introduction

For us a compact Klein surface of algebraic genus $g \geqslant 2$ will mean a pair (S, τ), where S is a closed Riemann surface of genus $g \geqslant 2$ and $\tau : S \to S$ is an anticonformal automorphism of S of order 2. In case that τ has no fixed points we say that it is an imaginary reflection; otherwise, we say that τ is a reflection. A compact Klein surface may also be seen as the quotient $R = S/\tau$. The surface S is called the complex double of R. Clearly, R is a closed surface of topological genus p if and only if τ is an imaginary reflection and S has genus $g = p - 1$. Generalities on Klein surfaces can be found, for instance, in [**4**]. If S is a closed Riemann surface, then we will denote by $\mathrm{Aut}^+(S)$ its group of conformal automorphisms and by $\mathrm{Aut}(S)$ its group of conformal and anticonformal automorphisms. The group $\mathrm{Aut}(S, \tau)$ of automorphisms of a compact Klein surface (S, τ) is by definition the subgroup of $\mathrm{Aut}(S)$ consisting of the those automorphisms that commutes with τ. If we set $\mathrm{Aut}^+(S, \tau) = \mathrm{Aut}(S, \tau) \cap \mathrm{Aut}^+(S)$, then we have that $\mathrm{Aut}(S, \tau)$ is generated by τ and $\mathrm{Aut}^+(S, \tau)$. Generalities on automorphisms on compact Klein surfaces may be found, for instance, in [**21, 24**]. If the genus of S is $g \geqslant 2$, then we have Hurwitz's bound $|\mathrm{Aut}^+(S)| \leqslant 84(g-1)$ [**14**]. It is well known that Hurwitz's bound is attained by an infinite number of values of g [**20**] and also that is not the case for infinite many other values of g. In particular,

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the above asserts that for a compact Klein surface (S, τ) of algebraic genus $g \geqslant 2$ we have that $|\mathrm{Aut}^+(S,\tau)| \leqslant 84(g-1)$. If τ is a reflection, then it is known that $|\mathrm{Aut}^+(S,\tau)| \leqslant 12(g-1)$ [**21**]. This bound is attained for infinitely many values of g (also not attained for infinitely many other values of g). If τ is an imaginary reflection, then there are infinitely many values of $g \geqslant 2$ for which there is a closed Klein surface (S,τ) of algebraic genus g for which $|\mathrm{Aut}^+(S,\tau)| = 84(g-1)$ [**24**]. If we set

$$v(g) = \mathrm{Max}|\mathrm{Aut}^+(S,\tau)|,$$

where (S,τ) runs over all closed non-orientable Klein surfaces of algebraic genus $g \geqslant 2$ (Max means maximum), then the results in [**7**] asserts that

$$v(g) \geqslant u(g) = \begin{cases} 4(g+1) & g \text{ even} \\ 8(g-1) & g \text{ odd} \end{cases}$$

and that there are infinitely many values of $g \geqslant 2$ for which we have equality in both cases (with the possible exception of $g \equiv 2 \mod 12$). The main porpoise of this note is to provide explicit examples of closed non-orientable Klein surfaces (S,τ), of algebraic genus $g \geqslant 2$, for which there is a subgroup H of $|\mathrm{Aut}^+(S,\tau)|$ with order at least $v(g)$ and so that the action of H can be reflected by a suitable Schottky uniformization of (S,τ). Of course theorem 3.2 in this note may be also obtained from the results of [**7**] and we do not claim in here that the result is new. Our idea is to give a different approach, the explicit construction of Schottky groups reflecting the action of the involved groups, and that is the novelty of this note. We also provide a construction of closed non-orientable surfaces of algebraic genus g, for infinite many values of integers $g \geqslant 2$, so that they admit a group of automorphisms of order $12(g-1)$ which can be reflected by Schottky groups (see theorem 3.3).

2. Schottky Uniformizations

The reader can find a good reference on the theory of Kleinian groups and Möbius transformations in [**23**].

A *Schottky group of genus* g is by definitions a Kleinian group G generated by loxodromic transformations, say $\alpha_1, \ldots, \alpha_g$, so that there are $2g$ disjoint simple loops, say $C_1, C_1', \ldots, C_g, C_g'$, all of them bounding a domain of connectivity $2g$, say $\mathcal{D} \subset \widehat{\mathbb{C}}$, such that

(1) $\alpha_j(C_j) = C_j'$, for $j = 1, ..., g$; and
(2) $\alpha_j(\mathcal{D}) \cap \mathcal{D} = \emptyset$, for $j = 1, \ldots, g$.

In the above, the set of loops, $C_1, \ldots, C_g'$ are called *defining loops* for the *Schottky generators* $\alpha_1, \ldots, \alpha_g$. It is well known that the region of discontinuity Ω of a Schottky group G of genus g is connected and dense in $\widehat{\mathbb{C}}$, and that $S = \Omega/G$ is a closed Riemann surface of genus g. We say that the surface S is uniformized by the Schottky group G. Moreover, if $C_1, \ldots, C_g'$ is a collection of defining loops for G and we denote by V_j the projection of C_j on S, then $V_1, \ldots, V_g$ is a set of pairwise disjoint homologically independent simple loops. A Schottky group of genus g is a free group of rank g so that every element of G, different of the identity, is loxodromic (we say that G is purely loxodromic) [**23**]. These properties characterize Schottky groups within the class of

Kleinian groups of the second kind (discrete groups of Möbius transformations with non-empty region of discontinuity) [**22, 6**]. As a consequence of the results of [**1**] we have that Schottky groups can be also characterized as those geometrically finite purely loxodromic Kleinian group with connected region of discontinuity. Retrosection theorem [**5, 17**] asserts that every closed Riemann surface S of genus $g \geqslant 1$ can be uniformized by a Schottky group of rank g. A *Schottky uniformization of a closed Riemann surface* S is a triple $(\Omega, G, P : \Omega \to S)$, where G is a Schottky group with region of discontinuity Ω and $P : \Omega \to S$ is a Galois covering with G as covering group. A *real Schottky group* G *of genus* g is by definition a Schottky group of genus g that keeps invariant some circle $C_G \subset \widehat{\mathbb{C}}$. In this case, the limit set of G is contained in C_G and the reflection τ_G on C_G commutes with every element of G. If $g \geqslant 2$, then, as the limit set is infinite, we have that C_G is unique. If Ω is the region of discontinuity of the real Schottky group G, with invariant circle C_G and τ_G the reflection on C_G, then we have that $S = \Omega/G$ is a closed Riemann surface admitting a reflection τ which is induced by τ_G. We say that the compact (bordered) Klein surface (S, τ) is uniformized by G. A result due to Köbe [**18**] asserts that each compact bordered Klein surface (S, τ) can be uniformized by a suitable real Schottky group G. A *Schottky uniformization of a Klein surface* (S, τ) is by definition a Schottky uniformization of S for which τ lifts. As a consequence of Köebes uniformization theorem [**18**], if (S, τ) is a compact Klein surface with τ a reflection, then there is a Schottky uniformization of it. If (S, τ) is a compact Klein surface with τ an imaginary reflection, then the existence of a Schottky uniformization of it is granted by quasiconformal deformation theory and the fact that the topological action of an imaginary reflection is rigid.

3. Schottky Type Automorphisms

Let us consider a closed Riemann surface S of genus $g \geqslant 2$ and a subgroup H of $\mathrm{Aut}(S)$. We say that H is of Schottky type if it is possible to find a Schottky uniformization of S, say $(\Omega, G, P : \Omega \to S)$, for which the group H lifts: for each $h \in H$ there is an automorphism $\widehat{h} : \Omega \to \Omega$ satisfying $P\widehat{h} = hP$. If we have a closed Riemann surface S of genus $g \geqslant 2$ and H a subgroup of $\mathrm{Aut}(S)$ which is of Schottky type, then we have the existence of a Schottky uniformization of S, say $(\Omega, G, P : \Omega \to S)$, for which H lifts. Since the region of discontinuity of the Schottky group G is known to be a domain of type O_{AD} [**3**], we have that each lifting $\widehat{h}$, for $h \in H$, is in fact the restriction of an extended Möbius transformation. Let us denote by K the group of (extended) Möbius transformations formed by all possible liftings of the elements of H by the covering $P : \Omega \to S$. It follows that K is a group of (extended) Möbius transformations which contains the Schottky group G as a normal subgroup of finite index; the index equal to the order of H. In particular, K is a finitely generated (extended) Kleinian group (by Ahlfors' finiteness theorem [**2**]) with Ω as region of discontinuity (in particular, K is a finitely generated function group [**23**]), $K/G = H$ and $\Omega/K = S/H$. If we denote by H^+ (respectively, K^+) the index two subgroup of H (respectively, K) consisting of its conformal automorphisms, then $S/H^+ = \Omega/K^+$ cannot have signature $(0, 3; n_1, n_2, n_3)$ (the Riemann sphere with exactly three branched values). In fact, due to a result of I. Kra [**19**] a function group uniformizing an orbifold of signature $(0, 3; n_1, n_2, n_3)$ must be a (triangular) Fuchsian group of the first kind, in particular, with disconnected region of discontinuity, a contradiction. It follows from this and Riemann-Hurwitz's formula that $|H^{+}| \leqslant 12(g-1)$ and if the equality holds, then S/H^{+} is a the Riemann sphere with exactly 4 branch values of orders 2, 2, 2, 3. Moreover, it has been shown in [**13**] that if H^{+} is a group of Schottky type of conformal automorphisms of a closed Riemann surface of genus $g \geqslant 2$ of order bigger than $4(g+1)$, then its order is of the form $4n(g-1)/(n-2)$, for some $n \geqslant 3$. In that case, the quotient S/H^{+} is the Riemann sphere with with exactly 4 branch values of orders 2, 2, 2 and n. In particular, if H^{+} has order bigger than $8(g-1)$, then it must have order exactly $12(g-1)$.

3.1. Schottky Type Automorphisms of Compact Klein Surfaces. Given a compact Klein surface (S,τ) and H a subgroup of $\mathrm{Aut}(S,\tau)$, we say that H is of Klein-Schottky type if there is a Schottky uniformization of (S,τ) for which H lifts, in other words, H is of Klein-Schottky type if and only if $\widetilde{H} = \langle H, \tau \rangle$ is of Schottky type. As a consequence, if the algebraic genus of (S,τ) is $g \geqslant 2$, then $\widetilde{H}$ has order at most $24(g-1)$, in particular, $\widetilde{H}^{+}$ has order at most $12(g-1)$. In the case that τ is a reflection, the following is a re-interpretation of Köebes uniformization theorem of real surfaces [**18**].

Theorem 3.1 (Koebe's Real Uniformization). *If (S,τ) is a bordered Klein surface (τ is a reflection), then Aut(S) is of Klein-Schottky type.*

Remark 3.1. As for a closed Klein surface (S,τ), of algebraic genus $g \geqslant 2$, there are examples for which $|\mathrm{Aut}^{+}(S,\tau)| > 12(g-1)$, the above result is not longer true in this class. In [**8**] we have found some necessary conditions in order for a cyclic group of automorphisms to be of Klein-Schottky type.

In [**13**] we have constructed infinitely many values of $g \geqslant 2$ for which there is a closed Riemann surface of genus g with a group of conformal automorphisms of Schottky type with maximum possible order $12(g-1)$. Necessary conditions for K to be of Schottky type are given in [**9**] for the case that K only contains conformal automorphisms and in [**8**] for the case that K contains anticonformal automorphisms. For the conformal situation we have that such necessary conditions in [**9**] are also sufficient for cyclic groups [**9**], Abelian groups [**10**], dihedral groups [**11**], the alternating groups $\mathcal{A}_4$, $\mathcal{A}_5$ and the symmetric group $\mathcal{S}_4$ [**12**]. In [**8**] was considered the anticonformal cyclic case. In the case (S,τ) is a closed Klein surfaces of algebraic genus $g \geqslant 2$, as already observed, we may have subgroups H of $\mathrm{Aut}^{+}(S,\tau)$ with order bigger than $12(g-1)$, in particular, not of Klein-Schottky type. Moreover, it may happen that the order of H is less than $12(g-1)$ and still not of Klein-Schottky type. As said in the introduction, the following can be obtained as consequences of the results in [**7**]. But our approach is different and is given in terms of explicit constructions of Schottky groups.

Theorem 3.2.

(1) *For each $g \geqslant 2$ we may find a closed Riemann surface S of genus $g \geqslant 2$ with a Schottky type subgroup H of Aut$^{+}(S)$ of order $2(g+1)$. If $g \geqslant 3$ is odd, then H can be found with order $8(g-1)$. Moreover, the surface may be chosen so*

that it admits a reflection τ, commuting with each automorphism of H so that $\widetilde{H} = \langle H, \tau \rangle$ is of Schottky type and so that S/τ is orientable.

(2) *For each $g \geqslant 2$ there is a bordered Klein surface (S, τ) of algebraic genus g with a Klein-Schottky type group $H < Aut^{+}(S, \tau)$ of order $4(g+1)$. If $g \geqslant 3$ is odd, then there is a bordered Klein surface (S, τ) of algebraic genus g with a Klein-Schottky type group $H < Aut^{+}(S, \tau)$ of order $8(g-1)$. Moreover, for each of these bordered Klein surfaces S/τ is orientable.*

(3) *For each integer $g \geqslant 2$ we have a closed Klein surface (S, τ) of algebraic genus g with a Klein-Schottky type group $H < Aut^{+}(S, \tau)$ of order $4(g+1)$. If $g \geqslant 3$ is odd, we have a closed Klein surface (S, τ) of algebraic genus g with a Klein-Schottky type group $H < Aut^{+}(S, \tau)$ of order $8(g-1)$.*

Remark 3.2. In the above, part (2) is just consequence of part (1).

There are infinitely many integers $g \geqslant 2$ for which there is a bordered Klein surface of algebraic genus g with group of automorphisms of order $12(g-1)$, then of Klein-Schottky type as consequence of Köbe's theorem 3.1. In the case of closed non-orientable Klein surfaces we have the following.

Theorem 3.3. *There are infinitely many integers $g \geqslant 2$ for which there is a closed Klein surface (S, τ) of algebraic genus g with a Schottky type group $H < Aut^{+}(S, \tau)$ of order $12(g-1)$.*

The above is proved by giving explicit construction of Schottky uniformizations. We provide an explicit example in genus $g = 2$ of a situation as in the above theorem 3.3, which will be needed later in the general construction.

Example 3.1. Let us consider the real line L_1, the line L_2 defined by 0 and $e^{\pi i/3}$, the unit circle $\mathcal{C}$ and a circle Σ orthogonal to both $\mathcal{C}$ and L_1 (with center in the positive real line) and disjoint from L_2. Let τ_1, τ_2, τ_3 and τ_4 the reflections on L_1, L_2, $\mathcal{C}$ and Σ, respectively. If $K_0 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$, then K_0 is a Kleinian group with connected region of discontinuity Ω so that Ω/K_0 is a closed disc with exactly 4 branch values in its borders, of orders $2, 2, 2, 3$. Let $W = \tau_2\tau_1$, $\eta_1 = \tau_4\tau_3\tau_1$, $\eta_2 = W\eta_1W^{-1}$ and $\eta_3 = W^{-1}\eta_1W$. Then the group $K_1 = \langle \eta_1, \eta_2, \eta_3 \rangle$ is a normal subgroup of K_0 of index 12 and Ω/K_1 is a closed non-orientable surface of topological genus $p = 3$. The index two normal subgroup K_1^{+}, generated by $A = \eta_2\eta_1$ and $B = \eta_3\eta_1$, is a Schottky group of genus two and normal in K_0. In this way, we have produced an example of a closed non-orientable Klein surface (S, τ) of algebraic genus 2, say $S = \Omega/K_1^{+}$ and τ induced by any of the elements of $K_1 - K_1^{+}$, with the Schottky type group $K_0^{+}/K_1^{+} < \mathrm{Aut}^{+}(S, \tau)$ of order 12 (the Schottky bound in $g = 2$).

4. Proof of Theorem 3.2

4.1. Case of order $4(g+1)$. Let $p \geqslant 3$ and $g = p - 1$. Consider the real line L_1, the line L_2 defined by 0 and $e^{\pi i/p}$, the unit circle $\mathcal{C}$ and a circle Σ orthogonal to both $\mathcal{C}$ and L_1 (with center in the positive real line) and disjoint from L_2. Let τ_1, τ_2, τ_3 and τ_4 the reflections on L_1, L_2, $\mathcal{C}$ and Σ, respectively. If $K_0 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$, then K_0 is a Kleinian group with connected region of discontinuity Ω so that Ω/K_0 is a closed disc with exactly 4 branch values in its borders, of orders $2, 2, 2, p$. We have already considered the case $p = 3$ in example 3.1. The group K_0 satisfies the following relations:

$$\tau_1^2 = \tau_2^2 = \tau_3^2 = \tau_4^2 = 1$$

$$(\tau_2\tau_1)^p = (\tau_3\tau_1)^2 = (\tau_4\tau_1)^2 = (\tau_3\tau_2)^2 = (\tau_3\tau_4)^2 = 1.$$

Set $W = \tau_2\tau_1$, $R = \tau_4\tau_3$, $T = \tau_3\tau_1$ and $\eta = \tau_4\tau_1\tau_3$. The transformation η is an imaginary reflection keeping invariant the circle Σ. In this way, K_0 also has generators:

$$W, T, R, \eta$$

with relations:

$$W^p = T^2 = R^2 = \eta^2 = 1$$

$$(WT)^2 = (RT)^2 = (\eta T)^2 = (\eta R)^2 = RWR\eta W\eta = 1.$$

Set $\eta_1 = \eta$, $\eta_{j+1} = W\eta_j W^{-1}$, for $j = 1, ..., p-1$. We see that η_j is an imaginary reflection keeping invariant the circle $W^{j-1}(\Sigma)$. The group K_1 generated by the involutions η_1,..., η_p is normal subgroup of K_0 and index $4p$, in fact, $K_0/K_1 \cong \mathbb{Z}/2\mathbb{Z} \times D_p$, where D_p denotes the dihedral group of order $2p$. The index two normal subgroup G of K_1, consisting of its conformal automorphisms, is a Schottky group of genus $g = p - 1$ generated by the transformations

$$A_1 = \eta\eta_2,\ A_2 = \eta\eta_3,\ ...,\ A_{p-1} = \eta\eta_p,$$

with a fundamental system of loops given by the circles

$$\Sigma_1 = W(\Sigma),\ \Sigma_1' = \eta(\Sigma_1),\ ...,\ \Sigma_{p-1} = W^{p-1}(\Sigma) \text{ and } \Sigma_{p-1}' = \eta(\Sigma_{p-1}).$$

We have a closed Riemann surface $S = \Omega/G$ together an imaginary reflection $\tau : S \to S$ induced by η_1. The closed non-orientable Klein surface (S, τ) has algebraic genus $g = p-1$ and the group of conformal automorphisms $H = K_0^+/K_1^+ < \mathrm{Aut}^+(S, \tau)$, of order $4p$, is of Schottky type. This gives us half of part (3) of the theorem. To obtain the half of part (1) of the theorem, we just need to observe that the reflection τ_3 descends to a reflection $\widehat{\tau} : S \to S$ which commutes with all the automorphisms in H, in particular, $H < \mathrm{Aut}^+(S, \widehat{\tau})$.

Remark 4.1. If in the above construction we set $S_1 = TR$, $S_{j+1} = WS_jW^{-1}$, for $j = 1, ..., p-1$, then we have that each conformal involution S_j keeps invariant the circle $W^{j-1}(\Sigma)$ and, in particular, the group K_2 is free generated (in the combination theorems sense [**23**]) by S_1,..., S_p. The group K_2 is a Whittaker group [**16**] of genus $g = p - 1$. Since $S_jS_1 = \eta_j\eta$, for each $j = 2, ..., p$, we see that G is a hyperelliptic Schottky group [**16**]. In particular, the Riemann surface S obtained in the above construction is a hyperelliptic Riemann surface with hyperelliptic involution induced by TR.

4.2. Case of order $8(g-1)$, where $g \geqslant 3$ odd. As before, we consider $p \geqslant 4$ even and set $g = p-1$. Consider the real line L_1, the line L_2 defined by 0 and $e^{\pi i/4}$, the unit circle $\mathcal{C}$ and a circle Σ orthogonal to both $\mathcal{C}$ and L_1 (with center in the positive real line) and disjoint from L_2. Let τ_1, τ_2, τ_3 and τ_4 the reflections on L_1, L_2, $\mathcal{C}$ and Σ, respectively. If $K_0 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$, then K_0 is a Kleinian group with connected region of discontinuity Ω so that Ω/K_0 is a closed disc with exactly 4 branch values in its borders, of orders $2, 2, 2, 4$. Set $W = \tau_2\tau_1$, $T = \tau_3\tau_1$ and $J = \tau_1\tau_4$. The index two normal subgroup $K = K_0^+$, consisting of the conformal automorphisms in K_0, is a geometrically finite Kleinian group generated by the transformations T, W and J. This group is in fact isomorphic to the direct product of a dihedral group of order four (the Klein group) and a dihedral group of order 8, amalgamated over $\mathbb{Z}/2\mathbb{Z}$ by use of Maskit's combination theorems. Since K_0 has no parabolic transformations, we have then that all non-loxodromic transformations are conjugated to either W, W^2, W^3, J, T, WT or JT. We have that $K_0 = \langle K, \tau_3 \rangle$ and that τ_3 commutes with every transformation of K. Let us write $g = 2q+1$ with q an integer greater or equal to 1, and consider the direct product group $M = P \times Q$, where P is generated by

$$x, w, j, t,$$

with relations:

$$x^q = w^4 = j^2 = t^2 = (wt)^2 = (jt)^2 = 1,$$
$$wxw^{-1} = x^{-1},\ tx = xt,\ jxj = x^{-1},\ jwj = xw,$$

and

$$Q = \langle u : u^2 = 1 \rangle.$$

It is not hard to see that P is a finite group of order $8(g-1)$, so M has order $16(g-1)$. The homomorphism $\rho : K_0 \to M$, defined by

$$\rho(W) = w,\ \rho(J) = j,\ \rho(T) = t,\ \rho(\tau_3) = u,$$

is surjective since $\rho(JWJW^{-1}) = x$. Let G be the kernel of such a homomorphism. We can see that necessarily $G \lhd K$ and that G has index $8(g-1)$ in K. As observed above, the elliptic elements of K are conjugated to either W, W^2, W^3, J, T, WT or JT. None of these transformations belongs to G and, as a consequence, G is torsion-free. As K has no parabolic transformations, we have that G is purely loxodromic. It follows that G is geometrically finite, purely loxodromic, Kleinian group with connected region of discontinuity, then a Schottky group. Let us denote by Ω the region of discontinuity of G (which is the same as for K), by $H = K/G$. We have the regular coverings

$$\pi_K : \Omega \to R = \Omega/K$$
$$\pi_G : \Omega \to S = \Omega/G$$
$$\pi_S : S \to R$$

so that $\pi_S\pi_G = \pi_K$. As $R = S/H$ is the Riemann sphere with exactly 4 branch values of orders 2, 2, 2 and 4, and H is a group of order $8(g-1)$, we have from Riemann-Hurwitz's formula that S has genus g. In particular, G is a Schottky group of genus g. The reflection τ_3 induces on the surface $S = \Omega/G$ a reflection τ which commutes with every element of K/G. This shows the second half of part (1). To get the second half of part (3), we need to observe that on the closed Riemann surface S the imaginary reflection $\widehat{\tau}$ induced by $W^2\tau_3$ commutes with every automorphism of H (this is simply consequence of the fact that w^2 commutes with every element in P). It follows that $(S,\widehat{\tau})$ is a closed Klein surface of algebraic genus g with a Klein-Schottky type group $H < \mathrm{Aut}^+(S,\widehat{\tau})$ of order $8(g-1)$.

REMARK 4.2. In this case we have that the closed Riemann surface S has two anticonformal involutions: (i) a reflection τ and (ii) an imaginary reflection $\widehat{\tau}$, satisfying the equality $\mathrm{Aut}^+(S,\tau) = \mathrm{Aut}^+(S,\widehat{\tau})$.

5. Proof of Theorem 3.3

A non-elementary Kleinian group K is called real if its limit set $\Lambda(K)$ is contained in some circle C (the image of the unit circle by some Möbius transformation). In particular, the reflection τ on C commutes with each element of K. The fact that K is non-elementary asserts that $\Lambda(K)$ has infinitely many points and, in particular, the circle C and τ are uniquely determined. Set $K_0 = \langle K, \tau\rangle$ and Ω the region of discontinuity of K (then also the region of discontinuity of K_0).

LEMMA 5.1. *Assume we have a Schottky group $G < K$ of genus $\gamma \geqslant 2$ as normal subgroup of K and index $q(\gamma-1)$. Assume also that we have a set of free generators A_1,..., A_γ of G so that, for $j = 1, ..., \gamma$, xA_jx^{-1} is a word in these generators of odd length for every $x \in K$. Set $\widehat{G}$ the group generated by the glide-reflections $B_1 = \tau A_1$,..., $B_\gamma = \tau A_\gamma$. Then*

- (i) *$\widehat{G}$ is a free group of rank γ, freely generated by B_1,..., B_γ;*
- (ii) *$\widehat{G}$ is a normal subgroup of K_0 of index $2q(\gamma-1)$;*
- (iii) *$S = \Omega/\widehat{G}$ is a closed non-orientable Klein surface of topological genus $p = 2\gamma$;*
- (iv) *The surface S has a group $H = K_0/\widehat{G}$ of automorphisms of Schottky type and order $q(p-2)$.*

PROOF. Let us consider an element $t \in K_0$, then $t = \tau x$, for some $x \in K$. We then have that $tB_jt^{-1} = \tau x\tau A_jx^{-1}\tau = \tau xA_jx^{-1}$. But, by our hypothesis, we know that xA_jx^{-1} is a word on odd length in A_1,..., A_γ, say $W(A_1, ..., A_\gamma)$. Then we have that $W(B_1, ..., B_\gamma) = \tau W(A_1, ..., A_\gamma)$. In particular, $tB_jt^{-1} = W(B_1, ..., B_\gamma)$, obtaining the normality of $\widehat{G}$ in K_0. In this way, normality of $\widehat{G}$ in K_0 asserts that Ω is also the region of discontinuity of $\widehat{G}$. The Schottky group G is a Schottky group keeping the circle C invariant. It follows that G is classical Schottky group for the set of generators A_1,..., A_γ. In particular, a fundamental domain of $\widehat{G}$ is given by a collection of 2γ pairwise disjoint circles, say C_1, C'_1,..., C_γ, C'_γ, each one orthogonal to C, bounding a common domain D of connectivity 2γ, so that $A_j(C_j) = C'_j$ and $A_j(D) \cap D = \emptyset$. Then we also have $B_j(C_j) = C'_j$ and $B_j(D) \cap D = \emptyset$. As for the case of Schottky group, one has that $\widehat{G}$ is a free group of rank γ, freely generated by B_1,..., B_γ. In this way we see that $S = \Omega/\widehat{G}$ is a closed non-orientable Klein surface of topological genus $p = 2\gamma$ admitting the group $H = K_0/\widehat{G}$ as group of automorphisms of Schottky type. The order of H is equal to the index of $\widehat{G}$ in K_0 which is equal to the index of G in K, in consequence, $2q(\gamma-1) = q(p-2)$. □

LEMMA 5.2. *Let us consider the Schottky group G and the free group $\widehat{G}$, freely generated by the transformations B_1,..., B_γ, as in lemma 5.1. Choose a positive odd integer $n \geqslant 3$ and consider the normal subgroup of $\widehat{G}$*

$$\widehat{G}_n = \langle t^n, [u, v] : t, u, v \in \widehat{G} \rangle,$$

where $[u, v] = uvu^{-1}v^{-1}$. We have that

(i) *$\widehat{G}_n$ is a normal subgroup of K_0;*
(ii) *$\widehat{G}_n$ has index n^γ in $\widehat{G}$ and index $2q(\gamma - 1)n^\gamma$ in K_0;*
(iii) *$\widehat{G}_n$ is a free group of rank $m = n^\gamma(\gamma - 1) + 1$;*
(iv) *$S = \Omega/\widehat{G}_n$ is a closed non-orientable Klein surface of topological genus $2m$;*
(v) *The Klein surface S admits the group of automorphisms $H = K_0/\widehat{G}_n$, which is of Schottky type and of order $(2m - 2)$.*

PROOF. The normality of $\widehat{G}_n$ in K_0 is clear. Since

$$\widehat{G}/\widehat{G}_n \cong \overset{\gamma}{\oplus} \mathbb{Z}/n\mathbb{Z},$$

we have that $\widehat{G}_n$ has index n^γ in $\widehat{G}$ and index $2q(\gamma - 1)n^\gamma$ in K_0. Since a subgroup of a free group of rank l and index a is a free group of rank $a(l - 1) + 1$, it follows that $\widehat{G}_n$ is a free group of rank $m = n^\gamma(\gamma - 1) + 1$. We choose a set of free generators of $\widehat{G}_n$, say C_1,..., C_m. Some of these generators must be glide-reflections; if not, all of them will be loxodromic and $\widehat{G}_n$ will be a Schottky group, in particular, containing only orientation preserving transformations, a contradiction to the fact that $B_1^n \in \widehat{G}_n$. We may assume that C_1,..., C_r are glide reflections and C_{r+1},..., C_m are loxodromic. It follows that $\widehat{G}_n$ uniformizes a closed non-orientable surface S homeomorphic to

$$\left(\overset{2r}{\#} \mathbb{R}P_2\right) \# \left(\overset{m-r}{\#} (\mathcal{S}^1 \times \mathcal{S}^1)\right),$$

where $\mathcal{S}^1$ denotes the unit circle and, in particular, that S is a closed non-orientable Klein surface of genus $2m$. The Klein surface S admits the group of automorphisms $H = K_0/\widehat{G}_n$ which is of Schottky type by the construction. The order of H is

$$|H| = [K_0 : \widehat{G}_n] = 2q(\gamma - 1)n^\gamma = 2q(m - 1) = q(2m - 2),$$

as required. □

5.1. Proof of Theorem 3.3. Let us start with the group $K_0 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$ as in the example 3.1 given at the end of section 2. In this case, we may apply the above two lemmas with $\tau = \tau_3$ and $K = K_0^+$.

Step 1. Let us construct a Schottky group G, of genus $\gamma = 3$, satisfying the following:

(i) G is a normal subgroup of K_0^+;
(ii) G has index 12 in K_0^+.
(iii) There is a set of free generators $A_1, ..., A_3$ for G so that, for every $x \in \{\tau_1, \tau_2, \tau_4\}$, we have that xA_jx^{-1} a word of odd length in these generators.

Set $A_1 = (\tau_4\tau_2)^3$, $A_2 = WA_1W^{-1}$, $A_3 = W^{-1}A_1W$, where (as before) $W = \tau_2\tau_1$. The group G generated by these three transformations is a Schottky group of genus 3 with fundamental domain bounded by the circles $\tau_3(W(\Sigma))$, $\tau_3(W^{-1}(\Sigma))$ and their translates by W and W^{-1}. In this case, we have the following relations:

$$\begin{cases} \tau_1 A_1 \tau_1 = A_3^{-1}; & \tau_1 A_2 \tau_1 = A_2^{-1}; \\ \tau_2 A_1 \tau_2 = A_1^{-1}; & \tau_2 A_2 \tau_2 = A_3^{-1}; \\ \tau_4 A_1 \tau_4 = A_1^{-1}; & \tau_4 A_2 \tau_4 = A_1 A_2 A_3; \quad \tau_4 A_3 \tau_4 = A_3^{-1} \end{cases}$$

As a consequence, we have that G satisfies (i) and (iii). The quotient K_0/G turns out to be a group of order 24, giving (ii).

Step 2. By step 1, the group G satisfies the properties needed by lemma 1, respect to $K = \langle K_0, \tau \rangle$. In particular, if Ω denotes the region of discontinuity of K_0, then $\Omega/\widehat{G}$ is a genus 6 closed non-orientable Klein surface with a group of automorphisms of Schottky type of maximum possible order 48.

Step 3. Once we have the free group $\widehat{G}$, freely generated by the glide-reflections B_1, B_2 and B_3, we may use lemma 2 for each odd positive integer $n \geqslant 3$ to get a group $\widehat{G}_n$. We have that $\widehat{G}_n$ is a normal subgroup of K_0 has index n^3 in $\widehat{G}$ and index $48n^3$ in K_0. It follows that $\widehat{G}_n$ is a free group of rank $m = 2n^3 + 1$. The closed Klein surface that $\widehat{G}_n$ uniformizes has topological genus $p_n = 2(2n^3 + 1)$ and a group of automorphisms of Schottky type $H = K_0/\widehat{G}_n$ of order

$$|H| = [K_0 : \widehat{G}_n] = 48n^3 = 12(g_n - 1),$$

where $p_n = g_n + 1$. If we denote by $G_n = \widehat{G}_n^+$ the index two subgroup of orientation preserving transformations in $\widehat{G}_n$, then we obtain that G_n is a Schottky group of genus g_n, uniformizing a closed Riemann surface S_n. On the surface S_n, of genus g_n, we have an imaginary reflection $\tau_n : S_n \to S_n$, induced by $\widehat{G}_n - G_n$. The quotient S_n/τ_n is uniformized by $\widehat{G}_n$ and we have that K_0^+/G_n is a conformal group of Schottky type and order $12(g_n - 1)$ as desired. □

Remark 5.1. Lemmas 5.1 and 5.2 and arguments similar to the constructions done above permit to construct infinite sequences of values of $g \geqslant 2$ for which there is a closed Klein surface (S, τ) of algebraic genus g with a Schottky type group of conformal automorphisms $H < \mathrm{Aut}^+(S, \tau)$ of order $q(g-1)$, for certain admissible values of q. For instance, for $q = 6$, let us consider a Fuchsian group $K = \langle A, B : A^3 = B^2 = 1 \rangle$ so that $\mathbb{H}^2/K$ is an open disc with exactly two branch values in its interior of orders 2 and 3. Let τ the reflection on the boundary circle of $\mathbb{H}^2 = \{z \in \mathbb{C} : |z| < 1\}$. Take the normal subgroup of K given by $G = \langle\langle (AB)^3 \rangle\rangle = \langle A_1, A_2, A_3 \rangle$, where $A_1 = (AB)^3$, $A_2 = AA_1A^{-1} = A^{-1}BABABA^{-1}$ and $A_3 = A^{-1}A_1A = (BA)^3$. We have that G is a Schottky group of genus 3 and $K/G \cong \mathcal{A}_4$, the alternating group of order 12. In this case, we have $BA_1B = A_3$, $BA_2B = A_1^{-1}A_2^{-1}A_3^{-1}$. It follows that we have the conditions of lemma 1 and, in this way, we get an infinite sequence of values of $g \geqslant 2$ for which there is a closed Klein surface of algebraic genus g admitting a Schottky type group of conformal automorphisms of order $6(g-1)$.

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