

Two closed-form evaluations for the generalized hypergeometric function ${}_4F_3\left(\frac{1}{16}\right)$

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ABSTRACT. The objective of this short note is to provide two closed-form evaluations for the generalized hypergeometric function ${}_4F_3$ of the argument $\frac{1}{16}$. This is achieved by means of separating a generalized hypergeometric function ${}_3F_2$ into even and odd components, together with the use of two known results for ${}_3F_2(\pm\frac{1}{4})$ available in the literature. As an application, we obtain an interesting infinite-sum representation for the number π^2 . Certain connections with the work of Ramanujan and other authors are discussed, involving other special functions and binomial sums of different kinds.

1. Introduction

The generalized hypergeometric function ${}_pF_q(z)$ with p numerator and q denominator parameters is defined by the series expansion

$$(1.1) \quad {}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z\right) = \sum_{n \geq 0} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{z^n}{n!},$$

where $(a)_n$ is the Pochhammer symbol [1] (shifted factorial, with $(1)_n = n!$),

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1) \dots (a+n-1) & n \in \mathbb{N}; \\ 1 & n = 0; \end{cases} \quad a \in \mathbb{C}.$$

As usual, p and q are non-negative integers, and the parameters $a_j \in \mathbb{C}$, $j = 1, \dots, p$ and $b_k \in \mathbb{C} \setminus \mathbb{Z}_0^-, k = 1, \dots, q$, where \mathbb{C} denotes the set of complex numbers, \mathbb{Z}_0^- is the set of negative integers along with zero.

When $p \leq q$, the power series (1.1) converges for all finite values of z with $|z| < \infty$. In the case $p = q+1$, it converges for $|z| < 1$ and diverges for $|z| > 1$. On the unit circle $|z| = 1$, the sufficient condition for convergence of (1.1) with $p = q+1$ is $\Re(s) > 0$

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where s is the parametric excess

$$s := \sum_{k=1}^q b_k - \sum_{j=1}^p a_j.$$

Also, when $p = q + 1$ and $\{|z| = 1, z \neq 1\}$, the series (1.1) converges with $-1 < \Re(s) \leq 0$. Analytic continuation can be employed to define the function ${}_pF_q(z)$ for z values outside the convergence region of the series (1.1).¹

For more details on generalized hypergeometric function ${}_pF_q(z)$ we refer to standard textbooks [3, 4, 5, 6, 7].

The subject of the present short communication are the special representatives of generalized hypergeometric functions belonging to the class $p = q + 1$ with specializations $p = 3$ and $p = 4$.

The main goal is the derivation of two new closed-form expressions for the function ${}_4F_3$ with specific argument $\frac{1}{16}$ starting from two known evaluations of ${}_3F_2(\pm\frac{1}{4})$. It is interesting that one of these last ${}_3F_2$ functions, ${}_3F_2(-\frac{1}{4})$, has been well-known in mathematics and appeared in the standard reference book by Prudnikov Brychkov and Marichev [8] as entry 7.4.6.1, while another one, ${}_3F_2(\frac{1}{4})$, resulted as a by-product in calculations of certain Feynman integrals by one of the authors in 2010 [9].

Our motivation consists of several components. First of all, (i) it is of primary interest to obtain explicit determinations of (any) hypergeometric functions. On the other hand, (ii) we wanted to bring the essentially unknown evaluations (3.2) and (5*) from a specific paper in theoretical physics [9] to the broad mathematical community. We also found it interesting to (iii) discuss some implications of the fusion of summation formulas (3.1) and (3.2), in particular, the infinite-sum representation of the number π^2 in (3.9).

Finally, it often happens that quite the same mathematical calculations are performed in very different and seemingly disconnected branches of theoretical physics and mathematics. And it is quite interesting and useful to establish connections between such differently motivated calculations. Thus, our last motivation's component was to (iv) perform a bright comparison of different approaches, functions and forms of results related to our initial input formulas (3.1) and (3.2). Thus it appeared that they were implicitly present in Ramanujan's notebooks, as well as in quite different modern papers in mathematics and theoretical physics. We hope that our "excursion" from Ramanujan to present days won't be no less interesting for the reader as our main ${}_4F_3$ results.

2. Even and odd components of ${}_pF_q(z)$

By decomposing the generalized hypergeometric function ${}_pF_q(z)$ into even and odd components and making use of standard identities for the Pochhammer symbol [1],

$$(a)_{2n} = 2^{2n} \left(\frac{a}{2}\right)_n \left(\frac{a}{2} + \frac{1}{2}\right)_n \quad \text{and} \quad (a)_{2n+1} = a 2^{2n} \left(\frac{a}{2} + \frac{1}{2}\right)_n \left(\frac{a}{2} + 1\right)_n,$$

¹Several interesting instances of the analytic continuation of the ${}_3F_2$ function are provided in [2].

one can easily obtain the following two general results, recorded, for example, in the reference book by Prudnikov et. al. [8, 7.2.3.42], viz.

$$(2.1) \quad \begin{aligned} & {}_{q+1}F_q \left(\begin{matrix} a_1, a_2, \dots, a_{q+1} \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right) + {}_{q+1}F_q \left(\begin{matrix} a_1, a_2, \dots, a_{q+1} \\ b_1, b_2, \dots, b_q \end{matrix} \middle| -z \right) \\ &= 2 {}_{2q+2}F_{2q+1} \left(\begin{matrix} \frac{a_1}{2}, \frac{a_1}{2} + \frac{1}{2}, \dots, \frac{a_{q+1}}{2}, \frac{a_{q+1}}{2} + \frac{1}{2} \\ \frac{1}{2}, \frac{b_1}{2}, \frac{b_1}{2} + \frac{1}{2}, \dots, \frac{b_q}{2}, \frac{b_q}{2} + \frac{1}{2} \end{matrix} \middle| z^2 \right) \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} & {}_{q+1}F_q \left(\begin{matrix} a_1, a_2, \dots, a_{q+1} \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right) - {}_{q+1}F_q \left(\begin{matrix} a_1, a_2, \dots, a_{q+1} \\ b_1, b_2, \dots, b_q \end{matrix} \middle| -z \right) \\ &= 2z \frac{a_1 a_2 \dots a_{q+1}}{b_1 b_2 \dots b_q} {}_{2q+2}F_{2q+1} \left(\begin{matrix} \frac{a_1}{2} + \frac{1}{2}, \frac{a_1}{2} + 1, \dots, \frac{a_{q+1}}{2} + \frac{1}{2}, \frac{a_{q+1}}{2} + 1 \\ \frac{3}{2}, \frac{b_1}{2} + \frac{1}{2}, \frac{b_1}{2} + 1, \dots, \frac{b_q}{2} + \frac{1}{2}, \frac{b_q}{2} + 1 \end{matrix} \middle| z^2 \right). \end{aligned}$$

3. Two closed-form evaluations for ${}_4F_3(\frac{1}{16})$

In this section, we employ the above procedure to the two following results:

$$(3.1) \quad {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| -\frac{1}{4} \right) = \frac{\pi^2}{10},$$

well known from [8, 7.4.6.1], and

$$(3.2) \quad {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \frac{1}{4} \right) = \frac{1}{2\sqrt{3}} \psi' \left(\frac{1}{3} \right) - \frac{\pi^2}{3\sqrt{3}},$$

obtained in [9, (26), (41)] in the course of Feynman-graph calculations. This leads us to two new closed-form summation formulas for the ${}_4F_3$ series with argument $\frac{1}{16}$, asserted in the following theorem.

THEOREM 1. *The following two closed-form evaluations for the function ${}_4F_3(\frac{1}{16})$ hold true:*

$$(3.3) \quad {}_4F_3 \left(\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \\ \frac{1}{2}, \frac{5}{4}, \frac{5}{4} \end{matrix} \middle| \frac{1}{16} \right) = \frac{1}{2} \left[\pi^2 \left(\frac{1}{10} - \frac{1}{3\sqrt{3}} \right) + \frac{1}{2\sqrt{3}} \psi' \left(\frac{1}{3} \right) \right]$$

and

$$(3.4) \quad {}_4F_3 \left(\begin{matrix} \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4} \\ \frac{3}{2}, \frac{7}{4}, \frac{7}{4} \end{matrix} \middle| \frac{1}{16} \right) = 36 \left[\frac{1}{2\sqrt{3}} \psi' \left(\frac{1}{3} \right) - \pi^2 \left(\frac{1}{10} + \frac{1}{3\sqrt{3}} \right) \right],$$

where $\psi(z)$ is the psi-function, the logarithmic derivative of the Euler gamma function $\Gamma(z)$.

PROOF. The derivation of the results (3.3) and (3.4) asserted by the Theorem 1 are straightforward and based upon identities (2.1) and (2.2). We set therein $q = 2$ and substitute $a_1 = a_2 = a_3 = \frac{1}{2}$ and $b_1 = b_2 = \frac{3}{2}$, along with $z = \frac{1}{4}$. After some simplifications in resulting hypergeometric ${}_6F_5$ functions, we make use of explicit expressions from (3.1) and (3.2) and arrive at closed-form evaluations (3.3) and (3.4). This completes the proof of the Theorem 1. \square

Application. The following new hypergeometric representation of π^2 can be easily obtained from (3.3) and (3.4) by subtracting the appropriately weighted equations, so that the terms containing the derivative of the ψ function disappear:

$$(3.5) \quad \frac{\pi^2}{10} = {}_4F_3\left(\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \\ \frac{1}{2}, \frac{5}{4}, \frac{5}{4} \end{matrix} \middle| \frac{1}{16}\right) - \frac{1}{72} {}_4F_3\left(\begin{matrix} \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4} \\ \frac{3}{2}, \frac{7}{4}, \frac{7}{4} \end{matrix} \middle| \frac{1}{16}\right).$$

Actually, the last formula directly reproduces the original result of (3.1) if we recognize here the difference of even and odd parts of ${}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}, \frac{1}{4})$. However, we can alternatively consider the coefficients of ${}_4F_3$ functions in (3.5) as

$$(3.6) \quad c_1(n) = \frac{\left(\frac{1}{4}\right)_n^3 \left(\frac{3}{4}\right)_n}{\left(\frac{5}{4}\right)_n^2 \left(\frac{1}{2}\right)_n n!} = \frac{2^{-2n} \Gamma\left(2n + \frac{1}{2}\right)}{(4n+1)^2 \Gamma\left(n + \frac{1}{2}\right) n!}$$

and

$$(3.7) \quad c_2(n) = \frac{\left(\frac{3}{4}\right)_n^3 \left(\frac{5}{4}\right)_n}{\left(\frac{7}{4}\right)_n^2 \left(\frac{3}{2}\right)_n n!} = 9 \frac{2^{-2n} \Gamma\left(2n + \frac{3}{2}\right)}{(4n+3)^2 \Gamma\left(n + \frac{3}{2}\right) n!},$$

and produce thereof the coefficients of the linear combination in (3.5) via

$$(3.8) \quad c(n) = c_1(n) - \frac{1}{72} c_2(n) = \frac{2^{-2n} (4n(4n(12n+29)+81)+71) \Gamma\left(2n + \frac{1}{2}\right)}{16n!(4n+1)^2(4n+3)^2 \Gamma\left(n + \frac{3}{2}\right)}.$$

Now, summing up the whole combination from the right-hand side of (3.5) with coefficients $c(n)$, we obtain the following representation of this identity in the form of the infinite single sum:

$$(3.9) \quad \frac{\pi^2}{10} = \frac{1}{16} \sum_{n=0}^{\infty} \frac{4n(4n(12n+29)+81)+71}{(4n+1)^2(4n+3)^2 n!} \frac{\Gamma\left(2n + \frac{1}{2}\right)}{\Gamma\left(n + \frac{3}{2}\right)} \left(\frac{1}{64}\right)^n.$$

We note that the convergence rate of the sum in (3.9) is essentially enhanced compared to that in (3.1): As $n \rightarrow \infty$, the terms in the last sum decay $\propto 4^{-n}$ times faster as in the original ${}_3F_2$ function in (3.1).

The results (3.3) through (3.9) have been verified using the MATHEMATICA software [10].

Remark 1. The method of splitting the generalized hypergeometric series (1.1) into its even and odd parts to generate new evaluations of higher-order hypergeometric functions is not new. It goes back at least to Krupnikov and Kölbig [11, p. 84] and Exton [12], and has been used recently in a number of publications, among them [13, 14, 15]. Other aspects of separation a power series into its even and odd components are considered in [16] and [17, Chap. 3].

4. Relation to Ramanujan's notebooks, other functions, binomial and harmonic sums

Remark 2. Due to the relation [18, (15)], [19, p. 329], [20, (1.10)]

$$(4.1) \quad \text{Cl}_2\left(\frac{\pi}{3}\right) = \frac{1}{2\sqrt{3}} \left[\psi'\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2 \right],$$

the results (3.2), (3.3) and (3.4) can be alternatively expressed in terms of the maximum value $\text{Cl}_2(\frac{\pi}{3})$ of the Clausen function $\text{Cl}_2(\theta)$ [21, Ch. 4], [20]

$$(4.2) \quad \text{Cl}_2(\theta) = - \int_0^\theta d\theta \ln \left| 2 \sin \frac{\theta}{2} \right| = \sum_{n \geq 1} \frac{\sin n\theta}{n^2} = \text{ImLi}_2(e^{i\theta}).$$

This function, along with its natural extensions, is very important in the theory of dilogarithm, $\text{Li}_2(z)$, and polylogarithm functions [21], and in computations of the Feynman integrals [22, 23].

Remark 3. A few results, equivalent to the one in (3.2), have appeared in the literature before [9, (26), (41)], though in quite different forms.

The first one goes back to Ramanujan, and appears in Example (iii) of Entry 16 in [24, p. 40] and [25, p. 264], as an evaluation of the binomial sum

$$(4.3) \quad \sum_{k \geq 0} \binom{2k}{k} \frac{1}{2^{4k+1}(2k+1)^2} = \frac{3\sqrt{3}}{4} \sum_{k \geq 0} \frac{1}{(3k+1)^2} - \frac{\pi^2}{6\sqrt{3}}$$

where

$$(4.4) \quad 2^{-2k} \binom{2k}{k} = \frac{1}{k!} \left(\frac{1}{2}\right)_k = 2^{-2k} \frac{(2k)!}{(k!)^2} = \frac{(2k-1)!!}{(2k)!!} := \mu_k$$

is the normalized binomial mid-coefficient (see e. g. [26, (2.2)] and related references). Indeed, using for the function ${}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{1}{4})$ from (3.2) its series definition (1.1) along with the first equality from (4.4) and the simple relation $(\frac{1}{2})_k / (\frac{3}{2})_k = (2k+1)^{-1}$ between Pochhammer symbols, we see that it matches the binomial sum on the left-hand side of (4.3) up to an extra factor of $\frac{1}{2}$ in (4.3). To match the right-hand sides of (3.2) and (4.3), we have to take into account that [7, 5.15.1], [27, 5.1.7.16]

$$\sum_{k \geq 0} \frac{1}{(3k+1)^2} = \frac{1}{9} \psi'\left(\frac{1}{3}\right).$$

The same result has been reproduced by Zucker [28, (2.13)] in the form

$$(4.5) \quad \sum_{k \geq 0} \binom{2k}{k} \frac{1}{4^{2k}(2k+1)^2} = \frac{3\sqrt{3}}{4} L_{-3}(2).$$

In notations of Zucker, $\frac{3\sqrt{3}}{4} L_{-3}(2)$ means the imaginary part of dilogarithm $\text{Li}_2(e^{i\pi/3})$ via [28, (1.11)] and [28, (1.8)]. Hence we see, by noticing (4.2) and (4.1), that his

equation (4.5) is tantamount just to (3.2) expressed as

$$(5^*) \quad {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \frac{1}{4}\right) = \text{Cl}_2\left(\frac{\pi}{3}\right).$$

As noticed after (3.2), this relation appeared in [9] as a result of a calculation of a Feynman integral. A slightly different treatment of the same integral in [29] yielded an alternative relation,

$$(4.6) \quad \frac{3\sqrt{\pi}}{8} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\psi(n+1)}{\Gamma(n+\frac{3}{2})} 4^{-n} + \frac{\pi\sqrt{3}}{6} (\gamma_E + \ln 3) = \frac{2}{\sqrt{3}} \text{Cl}_2\left(\frac{\pi}{3}\right) = \frac{1}{3} \psi'\left(\frac{1}{3}\right) - \frac{2\pi^2}{9},$$

which is obtained by combining (A13) and (A.15) of [29]². In (4.6), γ_E is the Euler constant, given by the limiting behavior of harmonic numbers³

$$(4.7) \quad H_n := \sum_{k=1}^n \frac{1}{k} = \gamma_E + \psi(n+1) \quad \text{via} \quad \gamma_E = \lim_{n \rightarrow \infty} (H_n - \ln n).$$

Using the first equation in (4.7) and [27, Annex 1.6]

$$\Gamma\left(n + \frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} 2^{-n} \frac{(2n+1)!}{n!},$$

we can rewrite (4.6) in a more instructive form:

$$(4.8) \quad \sum_{n=0}^{\infty} \frac{H_n}{\binom{2n}{n} (2n+1)} = \frac{8}{3\sqrt{3}} \text{Cl}_2\left(\frac{\pi}{3}\right) - \frac{2\pi}{3\sqrt{3}} \ln 3.$$

Now, combining (4.3), (4.5), (5*), and (4.8), we obtain a nice chain of equalities,

$$(4.9) \quad {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \frac{1}{4}\right) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{4^{-2k}}{(2k+1)^2} = \frac{3\sqrt{3}}{8} \sum_{k=0}^{\infty} \frac{H_k}{\binom{2k}{k} (2k+1)} + \frac{\pi}{4} \ln 3 = \text{Cl}_2\left(\frac{\pi}{3}\right),$$

relating the generalized hypergeometric function from (3.2) and (5*) with corresponding binomial and inverse binomial harmonic sums.

It seems that identifications of results (3.2) and (5*) in terms of a generalized hypergeometric function ${}_3F_2$ of argument $\frac{1}{4}$ appear only in equations (26) and (41) of reference [9].

Remark 4. Finally, we mention a similar connection of the ${}_3F_2$ function of argument $-\frac{1}{4}$ from (3.1) with the sum $f(x)$ considered by Ramanujan in Example (ii) of Entry 8 (see [24, p. 24], [25, p. 250]), and that from [28, (3.9)].

Similarly as before, we easily establish the equivalence of (3.1) with [28, (3.9)], namely

$$(4.10) \quad \sum_{k \geq 0} \binom{2k}{k} \frac{(-1)^k}{2^{4k} (2k+1)^2} = \frac{\pi^2}{10}.$$

²The reference to [29] given in [9] is incorrect: Ref. [13] in [9] has to be replaced by [29].

³For more related information, see e. g. [26, p. 147].

Moreover, Zucker [28, p. 101] notices the equivalence of his equation (3.9) with the special result $f(1/\sqrt{5}) = \pi^2/20$ for the Ramanujan's sum

$$(4.11) \quad f(x) = \sum_{k=1}^{\infty} \frac{h_k x^{2k-1}}{2k-1}, \quad \text{where} \quad h_k = \sum_{j=1}^k \frac{1}{2j-1}.$$

Hence, combining (3.1), (4.10), and (4.11), we may write

$$(4.12) \quad {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| -\frac{1}{4}\right) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-16)^{-k}}{(2k+1)^2} = 2 \sum_{k=1}^{\infty} \frac{h_k 5^{-k+\frac{1}{2}}}{2k-1} = \frac{\pi^2}{10}.$$

The relations of the odd harmonic number h_k to harmonic numbers H and digamma function ψ are given by [26, (2.3)], viz.

$$h_k = H_{2k} - \frac{1}{2} H_k = \frac{1}{2} \left[\psi\left(k + \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right) \right].$$

We are quite sure that the relations (4.9) and (4.12) deserve further analytical investigation, in particular towards generalizations to other arguments of ${}_3F_2$ functions, apart from $\pm\frac{1}{4}$.

5. Concluding remarks

In this note, we have provided two closed-form evaluations for the generalized hypergeometric function ${}_4F_3$ of the argument $\frac{1}{16}$. A combination of these results yields an interesting representation (3.9) of the transcendental number π^2 in terms of an infinite sum with an enhanced convergence rate.

There are many theoretical and practical applications of ${}_pF_q$ functions in different fields of mathematics [30, 31, 32, 33, 34], statistics and combinatorics [32, 35, 14], theoretical physics [36, 2, 37] and engineering [38, 39].⁴ Hopefully, our results could be potentially useful in some of these areas. To the best of our knowledge, the established summation formulas (3.3) and (3.4) did not appear in the literature before; we believe, they represent a definite contribution to the theory of generalized hypergeometric functions.

Moreover, we have established certain connections of functions under consideration with the work of Ramanujan and other authors, with other special functions, and with binomial sums of different kinds, some of them including harmonic numbers H_k and h_k .

A large number of results closely related to that of (3.3), (3.4) and (3.9) are under investigation and will be subject of a subsequent publication.

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⁴It is quite impossible to give an exhaustive list of relevant references here; our choice is motivated by scientific interests of the authors.

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