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Integration Formulas Involving Fibonacci and Lucas Numbers

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Abstract. We present a range of difficult integration formulas involving Fibonacci and Lucas numbers and trigonometric functions. These formulas are often expressed in terms of special functions like the dilogarithm and Clausen's function. We also prove complements of integral identities of Dilcher (2000) and Stewart (2022). Many of our results are based on a fundamental lemma dealing with differentiation of complex-valued Fibonacci (Lucas) functions.

1. Introduction

In a recent paper from 2022, Stewart [**8**] derived some appealing integral representations for Fibonacci numbers F_n and Lucas numbers L_n. For instance, he proved the representation [**8**, Theorem 2.1]

$$\frac{F_{kn}}{F_k} = \frac{n}{2^n}\int_{-1}^{1}\left(L_k + F_k x\sqrt{5}\right)^{n-1}dx \qquad (n,k\in\mathbb{N}). \tag{1.1}$$

The special case of this identity for $k = 1$ is also discussed in Stewart's paper [**9**] from 2023. Also, in 2015, Glasser and Zhou [**4**] worked out an explicit integral representation for F_n involving trigonometric functions. Indeed, the main result in their paper is the representation of the form

$$F_n = \frac{\alpha^n}{\sqrt{5}} - \frac{2}{\pi}\int_0^\infty \frac{\sin(x/2)}{x}\,\frac{\cos(nx) - 2\sin(nx)\sin(x)}{5\sin^2(x)+\cos^2(x)}\,dx, \tag{1.2}$$

where $\alpha = (1+\sqrt{5})/2$ is the golden ratio and $n \in \mathbb{N}_0$. Another representation is given by Andrica and Bagdasar in [**2**]. The last example for such representations comes from the paper by Dilcher [**3**] from 2000 where he showed (among others) that

$$F_{2n} = \frac{n}{2}\left(\frac{3}{2}\right)^{n-1}\int_0^\pi \left(1 + \frac{\sqrt{5}}{3}\cos(x)\right)^{n-1}\sin(x)\,dx. \tag{1.3}$$

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In this paper, we go in the same direction. However, we do not intend to prove explicit integral representations for Fibonacci and Lucas numbers, but instead we deal with integration formulas involving these sequences and combinations of trigonometric functions. We begin by proving the following complements of Stewart's and Dilcher's integral identities

$$\int_{-1}^{1}\left(L_k+F_kx\sqrt{5}\right)^{n-2}\left(F_k\sqrt{5}+L_kx\right)dx=\frac{2^n}{(n-1)\sqrt{5}}\left(\frac{L_{kn}}{F_k}-\frac{F_{kn}L_k}{nF_k^2}\right),\quad n\notin\{0,1\},\tag{1.4}$$

$$\int_{-1}^{1}\left(L_k+F_kx\sqrt{5}\right)^{n-2}x\,dx=\frac{2^n}{(n-1)\sqrt{5}F_k}\left(\frac{L_{(n-1)k}}{2}-\frac{F_{nk}}{nF_k}\right).\tag{1.5}$$

and

$$\begin{aligned}&\int_0^{\pi}\left(1+\frac{\sqrt{5}}{3}\cos x\right)^{n-1}\ln\left(1+\frac{\sqrt{5}}{3}\cos x\right)\sin x\,dx\\&\qquad=\frac{6}{n\sqrt{5}}\left(\frac{2}{3}\right)^nL_{2n}\ln\alpha+\left(-\frac{1}{n}+\ln\left(\frac{2}{3}\right)\right)\left(\frac{2}{3}\right)^n\frac{3}{n}F_{2n},\quad n\in\mathbb{Z}^+.\end{aligned}\tag{1.6}$$

Then, we prove a range of difficult integral identities of which we chose the following ones as a showcase:

$$\int_0^{\pi/2}\frac{\tan^2 x}{1+L_{2r}\tan^2 x+\tan^4 x}\,dx=\begin{cases}\dfrac{\pi}{2}\dfrac{1}{F_r\sqrt{5}(F_r\sqrt{5}+2)}, & \text{if } r \text{ is odd;}\\[2mm] \dfrac{\pi}{2}\dfrac{1}{L_r(L_r+2)}, & \text{if } r \text{ is even;}\end{cases}$$

$$\int_0^{\pi}\frac{x\sin^3 x}{\left(4+5F_{2r}^2\sin^2 x\right)^2}\,dx=-\frac{1}{10}\frac{\pi}{F_{4r}^2}+\frac{2\pi\sqrt{5}}{25}\frac{L_{4r}}{F_{4r}^3}r\ln\alpha,$$

$$\int_0^{\pi/2}\frac{x^2}{L_r^2+4+4L_r\cos(2x)}\,dx=\frac{1}{L_r^2-4}\left(\frac{\pi^3}{24}+\frac{\pi}{2}\operatorname{Li}_2\left(\frac{2}{L_r}\right)\right),\quad r\geqslant 2$$

and

$$\begin{aligned}&\int_0^{\pi}\frac{x^2\cos(3x)}{L_r^2-4\cos^2(2x)}\,dx\\&=\left(\frac{1}{L_r}-1\right)\frac{\pi}{2}\frac{\sqrt{\beta^r}}{1-\beta^r}\left(\operatorname{Li}_2\left(\sqrt{\beta^r}\right)-\operatorname{Li}_2\left(-\sqrt{\beta^r}\right)\right)\\&\quad+\left(\frac{1}{L_r}+1\right)\frac{\pi}{2}\frac{\sqrt{\beta^r}}{1+\beta^r}\left(\operatorname{Cl}_2\left(2\arctan\left(\sqrt{\beta^r}\right)\right)+\operatorname{Cl}_2\left(\pi-2\arctan\left(\sqrt{\beta^r}\right)\right)\right)\\&\quad+\left(\frac{1}{L_r}+1\right)\frac{\pi\sqrt{\beta^r}}{1+\beta^r}\arctan\left(\sqrt{\beta^r}\right)\ln\left(\sqrt{\beta^r}\right).\qquad r\geqslant 2\text{ even;}\end{aligned}$$

where $\operatorname{Cl}_2(x)$ is Clausen's function (see Lemma 5).

We note that most of the integrals derived in this paper cannot be evaluated by a Computer Algebra System. We checked with Maple, version 18. Even in the few cases where evaluation by a symbolic language is possible, the results are generally quite complicated and unwieldy. Finally, we think that some of the integrals derived in this paper are suitable for addition in the table by Gradshteyn and Ryzhik [**5**], for instance, in Section 3.61 dealing with rational functions of sines and cosines and trigonometric functions of multiple angles and in Section 4.38-4.41 focusing on logarithms and trigonometric functions.

Our paper is particularly inspired by the following identities of Lewin [**7**, p. 308–309, Identities (4) and (7)–(13)]:

$$\int_0^{\pi/2} \operatorname{Li}_2(-q^2\tan^2 x)\,dx = 2\pi \operatorname{Li}_2(-q), \quad q \geqslant 0, \tag{1.7}$$

$$\int_0^{\infty} \frac{\arctan(qx)}{1+x^2}\,dx = \frac{\pi^2}{8} - \frac{1}{2}\operatorname{Li}_2\left(\frac{1-q}{1+q}\right) + \frac{1}{2}\operatorname{Li}_2\left(-\frac{1-q}{1+q}\right), \tag{1.8}$$

$$\int_0^{\pi/2} \arctan(Q\csc x)\,dx = \frac{\pi^2}{4} - \operatorname{Li}_2\left(\sqrt{1+Q^2}-Q\right) + \operatorname{Li}_2\left(-\sqrt{1+Q^2}+Q\right), \tag{1.9}$$

$$\int_0^{\pi} x\arctan\left(\frac{2q}{1-q^2}\sin x\right)dx = \pi\operatorname{Li}_2(q) - \pi\operatorname{Li}_2(-q), \quad q^2<1. \tag{1.10}$$

$$\int_0^{\pi/2} \frac{x^2\,dx}{1-Q\cos(2x)} = \frac{1+q^2}{1-q^2}\left(\frac{\pi^3}{24} + \frac{\pi}{2}\operatorname{Li}_2(-q)\right), q^2<1, Q=\frac{2q}{1+q^2}, \tag{1.11}$$

$$\int_0^{\pi} \frac{x^2}{1-Q\cos^2 x}\,dx = \frac{1+q}{1-q}\left(\frac{\pi^3}{3} + \pi\operatorname{Li}_2(q)\right), \quad q<1, \quad Q=\frac{4q}{(1+q)^2}, \tag{1.12}$$

$$\int_0^{\pi} \frac{x^2\,dx}{1-Q\cos(2x)} = \frac{1+q^2}{1-q^2}\left(\frac{\pi^3}{3} + \pi\operatorname{Li}_2(q)\right), \quad q<1, \quad Q=\frac{2q}{1+q^2}, \tag{1.13}$$

$$\int_0^{\pi} \frac{x^2\cos x\,dx}{1-Q\cos(2x)} = -\pi\frac{1+q^2}{1-q}\frac{\operatorname{Li}_2\left(\sqrt{q}\right) - \operatorname{Li}_2\left(-\sqrt{q}\right)}{\sqrt{q}}, q<1, Q=\frac{2q}{1+q^2}. \tag{1.14}$$

Obviously, the common feature in all these results is the appearance of the dilogarithm $\operatorname{Li}_2(z)$ on one or both sides of the equations. This special function is defined by

$$\operatorname{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \qquad |z|<1.$$

We proceed with a definition of the Fibonacci numbers F_n and the Lucas numbers L_n, and with some lemmas which we be used later. Both sequences are defined, for $n \in \mathbb{Z}$, through the recurrence relations $F_n = F_{n-1} + F_{n-2}, n \geqslant 2$, with initial values $F_0 = 0, F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2, L_1 = 1$. For negative subscripts we have $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$. They possess the explicit formulas

(known as the Binet forms)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z},$$

with $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. For more information we refer to the books by Koshy [**6**] and Vajda [**11**].

Lemma 1. *If $z = 2\arctan(\beta^r/i^r)$ where r is an integer and i is the imaginary unit, then*

$$\cos z = \frac{F_r\sqrt{5}}{L_r}, \qquad \sin z = \frac{2i^r}{L_r}, \qquad \tan z = \frac{2i^r}{F_r\sqrt{5}}. \tag{1.15}$$

Proof. This is a consequence of the fact that if $z = 2\arctan(p/q)$, then

$$\cos z = \frac{q^2 - p^2}{q^2 + p^2}, \qquad \sin z = \frac{2pq}{q^2 + p^2}, \qquad \tan z = \frac{2pq}{q^2 - p^2}.$$

So, for instance,

$$\cos z = \frac{(-1)^r - \beta^{2r}}{(-1)^r + \beta^{2r}} = \frac{\alpha^r - \beta^r}{\alpha^r + \beta^r} = \frac{\sqrt{5}F_r}{L_r},$$

as $\alpha\beta = -1$. The remaining relations also follow immediately. □

Lemma 2. *Let $f(x)$ and $l(x)$ be the infinite times differentiable, complex-valued Fibonacci and Lucas functions defined by*

$$f(x) = \frac{\alpha^x - \beta^x}{\alpha - \beta}, \quad l(x) = \alpha^x + \beta^x, \quad x \in \mathbb{R}. \tag{1.16}$$

Then

$$f(x)\big|_{x=j\in\mathbb{Z}} = F_j, \quad l(x)\big|_{x=j\in\mathbb{Z}} = L_j; \tag{1.17}$$

and

$$\Re\left(\frac{d}{dx}f(x)\bigg|_{x=j\in\mathbb{Z}}\right) = \frac{L_j}{\sqrt{5}}\ln\alpha, \quad \Re\left(\frac{d}{dx}l(x)\bigg|_{x=j\in\mathbb{Z}}\right) = F_j\sqrt{5}\ln\alpha, \tag{1.18}$$

$$\Im\left(\frac{d}{dx}f(x)\bigg|_{x=j\in\mathbb{Z}}\right) = -\frac{\pi\beta^j}{\sqrt{5}}, \quad \Im\left(\frac{d}{dx}l(x)\bigg|_{x=j\in\mathbb{Z}}\right) = \pi\beta^j. \tag{1.19}$$

Proof. First, since β is negative, we write

$$\beta^x = (-\beta)^x \exp\left(i\pi\left(2m+1\right)x\right), \quad m \in \mathbb{Z},$$

so that

$$\frac{d}{dx}\beta^x = \beta^x\left(i\pi\left(2m+1\right) + \ln\left(-\beta\right)\right), \quad m \in \mathbb{Z}.$$

We have

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{1}{\alpha-\beta}\left(\frac{d}{dx}\alpha^x - \frac{d}{dx}\beta^x\right)\\
&= \frac{1}{\alpha-\beta}\left(\alpha^x\ln\alpha - \beta^x\ln(-\beta) - i\pi(2m+1)\beta^x\right)\\
&= \frac{1}{\alpha-\beta}\left(\alpha^x\ln\alpha + \beta^x\ln\alpha - \beta^x\ln\alpha - \beta^x\ln(-\beta) - i\pi(2m+1)\beta^x\right)\\
&= \frac{1}{\alpha-\beta}\left((\alpha^x+\beta^x)\ln\alpha - \beta^x\ln(-\alpha\beta) - i\pi(2m+1)\beta^x\right)\\
&= \frac{1}{\alpha-\beta}\left((\alpha^x+\beta^x)\ln\alpha - i\pi(2m+1)\beta^x\right).
\end{aligned}$$

The first identity in (1.18) and the first identity in (1.19) now follow upon taking real and imaginary parts. For the imaginary part, we used the principal value, $m = 0$.

The derivation of the second identity in (1.18) and the second identity in (1.19) proceeds along the same line. □

LEMMA 3. *If r is an integer, then*

$$1-\beta^{2r} = \begin{cases}\beta^r F_r\sqrt{5}, & r \text{ even};\\ -\beta^r L_r, & r \text{ odd};\end{cases}, \quad 1+\beta^{2r} = \begin{cases}\beta^r L_r, & r \text{ even};\\ -\beta^r F_r\sqrt{5}, & r \text{ odd}.\end{cases} \tag{1.20}$$

PROOF. Let r be even. Then,

$$1-\beta^{2r} = (-1)^r - \beta^{2r} = \beta^r(\alpha^r - \beta^r) = \beta^r\sqrt{5}F_r.$$

The other cases are proved in exactly the same manner. □

LEMMA 4. *If r is an integer, then*

$$\begin{gathered}
F_{2r}-1 = \begin{cases}F_{r-1}L_{r+1}, & r \text{ odd};\\ L_{r-1}F_{r+1}, & r \text{ even};\end{cases}, \quad F_{2r+1}-1 = \begin{cases}L_rF_{r+1}, & r \text{ odd};\\ F_rL_{r+1}, & r \text{ even};\end{cases},\\
L_{2r+1}-1 = \begin{cases}L_rL_{r+1}, & r \text{ odd};\\ 5F_rF_{r+1}, & r \text{ even}.\end{cases}
\end{gathered} \tag{1.21}$$

PROOF. Apply the Binet forms for F_n and L_n, respectively. □

LEMMA 5. *If $x > 0$, then*

$$\Re\operatorname{Li}_2(ix) = \frac{1}{4}\operatorname{Li}_2(-x^2) = \Re\operatorname{Li}_2(-ix), \quad [\mathbf{7}, \text{p.293, Identity (7)}], \tag{1.22}$$

and

$$\begin{aligned}
\Im\operatorname{Li}_2(ix) &= \arctan x\ln x + \frac{1}{2}\operatorname{Cl}_2(2\arctan x) + \frac{1}{2}\operatorname{Cl}_2(\pi - 2\arctan x)\\
&= -\Im\operatorname{Li}_2(-ix);
\end{aligned} \tag{1.23}$$

where Cl_2 *is Clausen's function defined by* [**7**, p.291] *:*

$$\operatorname{Cl}_2(y) = \sum_{n=1}^{\infty}\frac{\sin(ny)}{n^2} = -\int_0^y \ln|2\sin(\theta/2)|d\theta,$$

and having the functional relations

$$\operatorname{Cl}_2(\pi+\theta) = -\operatorname{Cl}_2(\pi-\theta), \tag{1.24}$$

$$\operatorname{Cl}_2(\theta) = -\operatorname{Cl}_2(2\pi-\theta), \tag{1.25}$$

$$\frac{1}{2}\operatorname{Cl}_2(2\theta) = \operatorname{Cl}_2(\theta) - \operatorname{Cl}_2(\pi-\theta); \tag{1.26}$$

with the special values

$$\operatorname{Cl}_2(n\pi) = 0, \quad n\in\mathbb{Z}^+, \tag{1.27}$$

and

$$\operatorname{Cl}_2(\pi/2) = G = -\operatorname{Cl}_2(3\pi/2), \tag{1.28}$$

where $G=\sum_{j=0}^{\infty}(-1)^j/(1+2j)^2$ *is Catalan's constant. For more information on these special functions see* [**10**].

Identity (1.23) follows from (see [**7**, p.292, Identity (1)]) the fact that

$$\Im\operatorname{Li}_2(re^{iy}) = \omega\ln r + \frac{1}{2}\operatorname{Cl}_2(2\omega) + \frac{1}{2}\operatorname{Cl}_2(2y) - \frac{1}{2}\operatorname{Cl}_2(2\omega+2y),$$

where

$$\tan\omega = \frac{r\sin y}{1-r\cos y}.$$

Lemma 6 ([**1**]). *If s is a positive integer, then*

$$\arctan(\beta^s) = \frac{1}{2}\arctan\left(\frac{2}{F_s\sqrt{5}}\right), \quad \textit{if } s \textit{ is even}, \tag{1.29}$$

and

$$\arctan(-\beta^s) = \frac{1}{2}\arctan\left(\frac{2}{L_s}\right), \quad \textit{if } s \textit{ is odd}. \tag{1.30}$$

2. Complements of the integral identities of Stewart and Dilcher

To illustrate the importance and broad applicability of Lemma 2 we now derive (1.4), (1.5) and (1.6).

Theorem 1. *For all integers $n\geqslant 2$ and $k\geqslant 1$ we have*

$$\int_{-1}^{1}\left(L_k+F_kx\sqrt{5}\right)^{n-2}\left(F_k\sqrt{5}+L_kx\right)dx = \frac{2^n}{(n-1)\sqrt{5}}\left(\frac{L_{kn}}{F_k}-\frac{F_{kn}L_k}{nF_k^2}\right) \tag{1.4}$$

and

$$\int_{-1}^{1}\left(L_k+F_kx\sqrt{5}\right)^{n-2}(1-x)\,dx = \frac{2^n}{(n-1)F_k\sqrt{5}}\left(-\beta^{(n-1)k}+\frac{F_{nk}}{nF_k}\right). \tag{2.1}$$

Proof. The Fibonacci function form of (1.1) is

$$\int_{-1}^{1}\left(l(t)+f(t)x\sqrt{5}\right)^{n-1}dx = \frac{f(tn)2^n}{nf(t)},$$

which by differentiating with respect to t gives

(2.2)
$$(n-1)\int_{-1}^{1}\left(l(t)+f(t)x\sqrt{5}\right)^{n-2}\left(\frac{d}{dt}l(t)+x\sqrt{5}\frac{d}{dt}f(t)\right)dx=\frac{2^n}{f(t)}\frac{d}{dt}f(nt)-\frac{2^n f(nt)}{nf(t)^2}\frac{d}{dt}f(t).$$

Evaluating (2.2) at $t=k$ and taking real parts using (1.17) and (1.18) and substituting

$$\Re\,\frac{d}{dt}f(tn)\bigg|_{t=k}=\frac{L_{kn}}{\sqrt{5}}\ln\alpha,\quad \Re\,\frac{d}{dt}l(t)\bigg|_{t=k}=F_k\sqrt{5}\ln\alpha,\quad \Re\,\frac{d}{dt}f(t)\bigg|_{t=k}=\frac{L_k}{\sqrt{5}}\ln\alpha,$$
$$f(tn)|_{t=k}=F_{kn},\quad f(t)|_{t=k}=F_k,\quad l(t)|_{t=k}=L_k,$$

we obtain

$$(n-1)\int_{-1}^{1}\left(L_k+F_kx\sqrt{5}\right)^{n-2}\left(F_k\sqrt{5}\ln\alpha+x\sqrt{5}\frac{L_k}{\sqrt{5}}\ln\alpha\right)dx$$
$$=\frac{2^n}{F_k}\frac{L_{nk}}{\sqrt{5}}\ln\alpha-\frac{2^nF_{nk}}{nF_k^2}\frac{L_k}{\sqrt{5}}\ln\alpha,$$

from which (1.4) follows.

Similarly, evaluating (2.2) at $t=k$ and taking imaginary parts using (1.17) and (1.19) and substituting

$$\Im\,\frac{d}{dt}f(tn)\bigg|_{t=k}=-\frac{\pi\beta^{nk}}{\sqrt{5}},\quad \Im\,\frac{d}{dt}l(t)\bigg|_{t=k}=\pi\beta^k,\quad \Im\,\frac{d}{dt}f(t)\bigg|_{t=k}=-\frac{\pi\beta^k}{\sqrt{5}},$$
$$f(tn)|_{t=k}=F_{kn},\quad f(t)|_{t=k}=F_k,\quad l(t)|_{t=k}=L_k,$$

we have

$$(n-1)\int_{-1}^{1}\left(L_k+F_kx\sqrt{5}\right)^{n-2}\left(\pi\beta^k+x\sqrt{5}\left(-\frac{\pi\beta^k}{\sqrt{5}}\right)\right)dx$$
$$=\frac{2^n}{F_k}\left(-\frac{\pi\beta^{nk}}{\sqrt{5}}\right)-\frac{2^nF_{nk}}{nF_k^2}\left(-\frac{\pi\beta^k}{\sqrt{5}}\right),$$

and hence (2.1) after dividing through by $\pi\beta^k$. □

COROLLARY 2. *For all integers $n\geqslant 2$ and $k\geqslant 1$ we have*

(1.5)
$$\int_{-1}^{1}\left(L_k+F_kx\sqrt{5}\right)^{n-2}x\,dx=\frac{2^n}{(n-1)\sqrt{5}F_k}\left(\frac{L_{(n-1)k}}{2}-\frac{F_{nk}}{nF_k}\right).$$

PROOF. Combine (1.1) with (2.1). □

The complement of Dilcher's identity is given in the next theorem.

THEOREM 3. *For all integers $n\geqslant 1$, we have*

(1.6)
$$\int_0^{\pi}\left(1+\frac{\sqrt{5}}{3}\cos x\right)^{n-1}\ln\left(1+\frac{\sqrt{5}}{3}\cos x\right)\sin x\,dx$$
$$=\frac{6}{n\sqrt{5}}\left(\frac{2}{3}\right)^nL_{2n}\ln\alpha+\left(-\frac{1}{n}+\ln\left(\frac{2}{3}\right)\right)\left(\frac{2}{3}\right)^n\frac{3}{n}F_{2n}.$$

PROOF. Differentiating the Fibonacci function form of (1.3), that is,

$$\int_0^\pi \left(1+\frac{\sqrt{5}}{3}\cos x\right)^{t-1}\sin x\,dx=\frac{2f(2t)}{t}\left(\frac{2}{3}\right)^{t-1}$$

with respect to t gives

$$\begin{aligned}\int_0^\pi &\left(1+\frac{\sqrt{5}}{3}\cos x\right)^{t-1}\ln\left(1+\frac{\sqrt{5}}{3}\cos x\right)\sin x\,dx \\ &=4\left(\frac{2}{3}\right)^{t-1}\frac{d}{dt}f(2t)-2\frac{f(2t)}{t^2}\left(\frac{2}{3}\right)^{t-1}+\frac{2f(t)}{t}\left(\frac{2}{3}\right)^{t-1}\ln\left(\frac{2}{3}\right).\end{aligned}\tag{2.3}$$

Evaluating (2.3) at $t=n$ and taking the real part gives

$$\begin{aligned}\int_0^\pi &\left(1+\frac{\sqrt{5}}{3}\cos x\right)^{n-1}\ln\left(1+\frac{\sqrt{5}}{3}\cos x\right)\sin x\,dx \\ &=\frac{4}{n}\left(\frac{2}{3}\right)^{n-1}\Re\left.\frac{d}{dt}f(2t)\right|_{t=n}-2\frac{F_{2n}}{n^2}\left(\frac{2}{3}\right)^{n-1}+\frac{2F_{2n}}{n}\left(\frac{2}{3}\right)^{n-1}\ln\left(\frac{2}{3}\right) \\ &=\frac{4}{n}\left(\frac{2}{3}\right)^{n-1}\frac{L_{2n}}{\sqrt{5}}\ln(\alpha)-2\frac{F_{2n}}{n^2}\left(\frac{2}{3}\right)^{n-1}+\frac{2F_{2n}}{n}\left(\frac{2}{3}\right)^{n-1}\ln\left(\frac{2}{3}\right),\end{aligned}$$

which simplifies to (1.6). □

3. Results associated with (1.7)

THEOREM 4. *Let r be an integer. Then*

$$\int_0^{\pi/2}\ln\left(1+L_{2r}\tan^2 x+\tan^4 x\right)dx=\begin{cases}\pi\ln(F_r\sqrt{5}+2), & \text{if } r \text{ is odd;}\\ \pi\ln(L_r+2), & \text{if } r \text{ is even;}\end{cases}\tag{3.1}$$

$$\int_0^{\pi/2}\ln\left(\frac{\left(1+\alpha^{2r}+\tan^2 x\right)^2}{1+L_{2r}\tan^2 x+\tan^4 x}\right)dx=\begin{cases}\pi r\ln\alpha, & \text{if } r \text{ is odd;}\\ \pi\ln\left(\frac{(1+\alpha^r)^2}{L_r+2}\right), & \text{if } r \text{ is odd.}\end{cases}\tag{3.2}$$

PROOF. Differentiate (1.7) with respect to q to get

$$\int_0^{\pi/2}\ln\left(1+q^2\tan^2 x\right)dx=\pi\ln(1+q).\tag{3.3}$$

Set $q=\alpha^r$ and $q=-\beta^r$, in turn, for the case when r is an odd integer. Use $q=\alpha^r$ and $q=\beta^r$, in turn, for the case when r is an even integer. Combine according to the Binet formulas; addition gives (3.1) while subtraction gives (3.2). □

COROLLARY 5. *If r is an integer, then*

$$\int_0^{\pi/2}\frac{\tan^2 x}{1+L_{2r}\tan^2 x+\tan^4 x}\,dx=\begin{cases}\dfrac{\pi}{2}\dfrac{1}{F_r\sqrt{5}(F_r\sqrt{5}+2)}, & \text{if } r \text{ is odd;}\\ \dfrac{\pi}{2}\dfrac{1}{L_r(L_r+2)}, & \text{if } r \text{ is even.}\end{cases}\tag{3.4}$$

PROOF. Differentiate the Fibonacci and Lucas function form of (3.1) with respect to r, making use of (1.18). □

COROLLARY 6. *If r is an integer, then*

(3.5)
$$\int_0^{\pi/2} \frac{1}{1 + L_{2r}\tan^2 x + \tan^4 x}\,dx = \begin{cases} \frac{\pi}{2}\frac{1}{L_{2r}(\sqrt{5}F_r+2)}(L_{2r} + \sqrt{5}F_r - \frac{2}{\sqrt{5}F_r}), & \text{if } r \text{ is odd;} \\ \frac{\pi}{2}\frac{1}{L_{2r}(L_r+2)}(L_{2r} + L_r - \frac{2}{L_r}), & \text{if } r \text{ is even.} \end{cases}$$

PROOF. Replacing q by $1/q$ in (3.3) shows that

$$\int_0^{\pi/2} \ln\left(q^2 + \tan^2 x\right)\,dx = \pi\ln(1+q). \tag{3.6}$$

This yields

$$\int_0^{\pi/2} \frac{1}{q^2 + \tan^2 x}\,dx = \frac{\pi}{2q(1+q)}. \tag{3.7}$$

From here, we can proceed like in the proof of Theorem 4 getting

$$\int_0^{\pi/2} \frac{L_{2r} + 2\tan^2 x}{1 + L_{2r}\tan^2 x + \tan^4 x}\,dx = \frac{\pi}{2}\frac{\sqrt{5}F_r + L_{2r}}{\sqrt{5}F_r + 2}, \qquad r \text{ odd}$$

and

$$\int_0^{\pi/2} \frac{L_{2r} + 2\tan^2 x}{1 + L_{2r}\tan^2 x + \tan^4 x}\,dx = \frac{\pi}{2}\frac{L_r + L_{2r}}{L_r + 2}, \qquad r \text{ even.}$$

This completes the proof. □

LEMMA 7. *If n is a non-negative integer and q is a positive number, then*

(3.8)
$$\int_0^{\pi/2} \frac{dx}{(q^2 + \tan^2 x)^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{q^{2k}}\binom{n}{2k}\frac{(-1)^k}{2k+1}\tan^{2k} x = \frac{\pi}{2}\frac{1}{n+1}\left(\frac{1}{q^{2n+1}} - \frac{q}{(q(q+1))^{n+1}}\right).$$

PROOF. Differentiate (3.7) n times. □

THEOREM 7. *If n is a non-negative integer and r is a positive integer, then*

$$\begin{aligned} &\int_0^{\pi/2} \frac{dx}{(L_r^2 + 5F_r^2\tan^2 x)^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\frac{(-1)^k}{2k+1}\left(\frac{5F_r^2}{L_r^2}\right)^k \tan^{2k} x \\ &\qquad = \frac{\pi}{2}\frac{1}{n+1}\frac{1}{L_r^n F_r\sqrt{5}}\left(\frac{1}{L_r^{n+1}} - \frac{1}{(2\alpha^r)^{n+1}}\right) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\int_0^{\pi/2} \frac{dx}{(5F_r^2 + L_r^2\tan^2 x)^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\frac{(-1)^k}{2k+1}\left(\frac{L_r^2}{5F_r^2}\right)^k \tan^{2k} x \\ &\qquad = \frac{\pi}{2}\frac{1}{n+1}\frac{1}{(F_r\sqrt{5})^n L_r}\left(\frac{1}{(F_r\sqrt{5})^{n+1}} - \frac{1}{(2\alpha^r)^{n+1}}\right). \end{aligned} \tag{3.10}$$

PROOF. Use $q = L_r/F_r\sqrt{5}$ and $q = F_r\sqrt{5}/L_r$ in (3.8). □

In particular, we mention the special cases

$$\int_0^{\pi/2} \frac{dx}{(L_r^2 + 5F_r^2\tan^2 x)^2} = \frac{\pi}{4}\frac{1}{F_{2r}\sqrt{5}}\left(\frac{1}{L_r^2} - \frac{1}{4\alpha^{2r}}\right) \tag{3.11}$$

$$\int_0^{\pi/2} \frac{dx}{(5F_r^2 + L_r^2\tan^2 x)^2} = \frac{\pi}{4}\frac{1}{F_{2r}\sqrt{5}}\left(\frac{1}{5F_r^2} - \frac{1}{4\alpha^{2r}}\right) \tag{3.12}$$

with the special values

$$\int_0^{\pi/2} \frac{dx}{(1 + 5\tan^2 x)^2} = \frac{\pi\alpha}{16}$$

and

$$\int_0^{\pi/2} \frac{dx}{(5 + \tan^2 x)^2} = \frac{\pi}{400}\left(2 + \frac{7}{\alpha^2}\right).$$

THEOREM 8. *If n is a non-negative integer and r is any integer, then*

$$\begin{aligned}\int_0^{\pi/2} &\frac{dx}{(1+3\tan^2 x+\tan^4 x)^{n+1}} \\ &\times\left(\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{(-1)^k\tan^{2k}x}{2k+1}\sum_{j=0}^{n+1}\binom{n+1}{j}\tan^{2j}xL_{2n+2k-2j+r+2}\right) \\ &= \frac{\pi}{2}\frac{1}{n+1}\left(F_{2n+r+1}\sqrt{5} - \alpha^{r-1} + (-1)^{n+1}\beta^{3n+r+2}\right)\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\int_0^{\pi/2} &\frac{dx}{(1+3\tan^2 x+\tan^4 x)^{n+1}} \\ &\times\left(\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\frac{(-1)^k\tan^{2k}x}{2k+1}\sum_{j=0}^{n+1}\binom{n+1}{j}\tan^{2j}xF_{2n+2k-2j+r+2}\right) \\ &= \frac{\pi}{2\sqrt{5}}\frac{1}{n+1}\left(L_{2n+r+1} - \alpha^{r-1} + (-1)^{n}\beta^{3n+r+2}\right).\end{aligned} \tag{3.14}$$

PROOF. Use $q = \alpha$ and $q = -\beta$ in (3.8) and combine the resulting identities in accordance with the Binet formulas. □

In particular,

$$\int_0^{\pi/2} \frac{L_r\tan^2 x + L_{r+2}}{(1+3\tan^2 x+\tan^4 x)}\,dx = \frac{\pi}{2}\left(F_{r+1}\sqrt{5} - \alpha^{r-1} - \beta^{r+2}\right), \tag{3.15}$$

$$\int_0^{\pi/2} \frac{F_r\tan^2 x + F_{r+2}}{(1+3\tan^2 x+\tan^4 x)}\,dx = \frac{\pi}{2\sqrt{5}}\left(L_{r+1} - \alpha^{r-1} + \beta^{r+2}\right), \tag{3.16}$$

with the special values

$$\int_0^{\pi/2} \frac{1-\tan^2 x}{1+3\tan^2 x+\tan^4 x}\,dx = -\frac{\pi\beta^3}{2},$$
$$\int_0^{\pi/2} \frac{dx}{1+3\tan^2 x+\tan^4 x} = \frac{\pi\beta^2}{\sqrt{5}},$$
$$\int_0^{\pi/2} \frac{\tan^2 x}{1+3\tan^2 x+\tan^4 x}\,dx = -\frac{\pi\beta^3}{2\sqrt{5}}.$$

Remark. *Setting* $q=\alpha^r$, $q=\pm\beta^r$ *in* (3.8) *will deliver more identities of this nature.*

4. Results associated with (1.8)

Theorem 9. *If* m *is a non-negative integer and* r *is a positive integer, then*

$$\begin{aligned}\int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)(1+F_{2r}x^2)^{m+1}} &= \int_0^\infty \frac{x\,dx}{(1+x^2)(F_{2r}+x^2)^{m+1}}\\ &= \begin{cases} -\frac{1}{2}\sum_{j=1}^m \frac{1}{j}\frac{1}{F_{2r}^j(F_{r-1}L_{r+1})^{m-j+1}} + \frac{1}{2}\frac{\ln F_{2r}}{(F_{r-1}L_{r+1})^{m+1}}, & \text{if } r \text{ is odd, } r\neq 1;\\ -\frac{1}{2}\sum_{j=1}^m \frac{1}{j}\frac{1}{F_{2r}^j(L_{r-1}F_{r+1})^{m-j+1}} + \frac{1}{2}\frac{\ln F_{2r}}{(L_{r-1}F_{r+1})^{m+1}}, & \text{if } r \text{ is even;}\end{cases}\end{aligned} \tag{4.1}$$

$$\begin{aligned}\int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)(1+F_{2r+1}x^2)^{m+1}} &= \int_0^\infty \frac{x\,dx}{(1+x^2)(F_{2r+1}+x^2)^{m+1}}\\ &= \begin{cases} -\frac{1}{2}\sum_{j=1}^m \frac{1}{j}\frac{1}{F_{2r+1}^j(L_rF_{r+1})^{m-j+1}} + \frac{1}{2}\frac{\ln F_{2r+1}}{(L_rF_{r+1})^{m+1}}, & \text{if } r \text{ is odd;}\\ -\frac{1}{2}\sum_{j=1}^m \frac{1}{j}\frac{1}{F_{2r+1}^j(F_rL_{r+1})^{m-j+1}} + \frac{1}{2}\frac{\ln F_{2r+1}}{(F_rL_{r+1})^{m+1}}, & \text{if } r \text{ is even;}\end{cases}\end{aligned} \tag{4.2}$$

$$\begin{aligned}\int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)(1+L_{2r+1}x^2)^{m+1}} &= \int_0^\infty \frac{x\,dx}{(1+x^2)(L_{2r+1}+x^2)^{m+1}}\\ &= \begin{cases} -\frac{1}{2}\sum_{j=1}^m \frac{1}{j}\frac{1}{L_{2r+1}^j(L_rL_{r+1})^{m-j+1}} + \frac{1}{2}\frac{\ln L_{2r+1}}{(L_rL_{r+1})^{m+1}}, & \text{if } r \text{ is odd;}\\ -\frac{1}{2}\sum_{j=1}^m \frac{1}{j}\frac{1}{L_{2r+1}^j(5F_rF_{r+1})^{m-j+1}} + \frac{1}{2}\frac{\ln L_{2r+1}}{(5F_rF_{r+1})^{m+1}}, & \text{if } r \text{ is even;}\end{cases}\end{aligned} \tag{4.3}$$

Proof. Differentiating (1.8) with respect to q gives

$$\int_0^\infty \frac{x\,dx}{(1+x^2)(1+qx^2)} = -\frac{1}{2}\frac{\ln q}{1-q}$$

which writing $1/q$ for q also means

$$\int_0^\infty \frac{x\,dx}{(1+x^2)(q+x^2)} = -\frac{1}{2}\frac{\ln q}{1-q};$$

so that

$$\int_0^\infty \frac{x\,dx}{(1+x^2)(1+qx^2)} = -\frac{1}{2}\frac{\ln q}{1-q} = \int_0^\infty \frac{x\,dx}{(1+x^2)(q+x^2)}. \tag{4.4}$$

Differentiating (4.4) m times with respect to q gives

$$\begin{aligned}\int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)(1+qx^2)^{m+1}} &= \int_0^\infty \frac{x\,dx}{(1+x^2)(q+x^2)^{m+1}} \\ &= \frac{1}{2}\sum_{j=1}^m \frac{(-1)^{m-j}}{j}\frac{1}{q^j(1-q)^{m-j+1}} + \frac{(-1)^{m-1}}{2}\frac{\ln q}{(1-q)^{m+1}}.\end{aligned} \tag{4.5}$$

Using $q = F_{2r}$, $q = F_{2r+1}$ and $q = L_{2r+1}$ in turn in (4.5) while making use of (1.21) produces (4.1), (4.2) and (4.3). $\square$

THEOREM 10. *If r is a non-zero integer, then*

$$\begin{aligned}\int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)(L_r^2+5F_r^2x^2)^{m+1}} &= \int_0^\infty \frac{x\,dx}{(1+x^2)(5F_r^2+L_r^2x^2)^{m+1}} \\ &= \frac{1}{2^{2m+3}}\left(\sum_{j=1}^m \frac{(-1)^{r+(r+1)(m-j)}}{j}\left(\frac{4}{5F_r}\right)^j + (-1)^{(r+1)(m+1)}\ln\left(\frac{5F_r^2}{L_r^2}\right)\right).\end{aligned} \tag{4.6}$$

PROOF. Set $q = 5F_r^2/L_r^2$ in (4.5) and use the identity $L_n^2 = 5F_n^2 + (-1)^n 4$. $\square$

THEOREM 11. *If r is a non-zero even integer, then*

$$\begin{aligned}\int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)(L_r^2+4x^2)^{m+1}} &= \int_0^\infty \frac{x\,dx}{(1+x^2)(4+L_r^2x^2)^{m+1}} \\ &= \frac{1}{2(5F_r^2)^{m+1}}\left(\sum_{j=1}^m \frac{(-1)^{m-j}}{j}\left(\frac{5F_r^2}{4}\right)^j + (-1)^{m-1}\ln\left(\frac{4}{L_r^2}\right)\right).\end{aligned} \tag{4.7}$$

PROOF. Set $q = 4/L_r^2$ in (4.5) and use the identity $L_n^2 = 5F_n^2 + (-1)^n 4$. $\square$

THEOREM 12. *If r is a positive odd integer, then*

$$\begin{aligned}\int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)(5F_r^2+4x^2)^{m+1}} &= \int_0^\infty \frac{x\,dx}{(1+x^2)(4+5F_r^2x^2)^{m+1}} \\ &= \frac{1}{2L_r^{2(m+1)}}\left(\sum_{j=1}^m \frac{(-1)^{m-j}}{j}\left(\frac{L_r^2}{4}\right)^j + (-1)^{m-1}\ln\left(\frac{4}{5F_r^2}\right)\right).\end{aligned} \tag{4.8}$$

PROOF. Set $q = 4/(5F_r^2)$ in (4.5) and use the identity $L_n^2 = 5F_n^2 + (-1)^n 4$. $\square$

THEOREM 13. *If r is a positive integer, then*

$$\begin{aligned}\int_0^\infty \frac{x^{2m+1}\,dx}{(1+x^2)(1+F_{4r+1}x^2)^{m+1}} &= \int_0^\infty \frac{x\,dx}{(1+x^2)(F_{4r+1}+x^2)^{m+1}} \\ &= \frac{1}{2(F_{2r}L_{2r+1})^{m+1}}\left(\ln F_{4r+1} - \sum_{j=1}^m \frac{1}{j}\left(\frac{F_{2r}L_{2r+1}}{F_{4r+1}}\right)^j\right).\end{aligned} \tag{4.9}$$

PROOF. Set $q = F_{4r+1}$ in (4.5) and use the identity $F_{4n+1} - 1 = F_{2n}L_{2n+1}$. $\square$

5. Results associated with (1.9)

LEMMA 8. *If* $0 < q \leqslant 1$ *then*

$$\int_0^{\pi/2} \ln\left(\frac{\dfrac{1+q^2}{2q} + \sin x}{\dfrac{1+q^2}{2q} - \sin x}\right) dx = 2\,\mathrm{Cl}_2\left(2\arctan q\right) + 2\,\mathrm{Cl}_2\left(\pi - 2\arctan q\right) + 4\arctan q \ln q. \tag{5.1}$$

PROOF. Replace q by $i(1+q^2)/(2q)$ in (1.9) and take the real part. □

THEOREM 14. *If r is an even integer, then*

$$\int_0^{\pi/2} \ln\left(\frac{L_r + 2\sin x}{L_r - 2\sin x}\right) dx = 2\,\mathrm{Cl}_2\left(\arctan\left(\frac{2}{F_r\sqrt{5}}\right)\right) + 2\,\mathrm{Cl}_2\left(\pi - \arctan\left(\frac{2}{F_r\sqrt{5}}\right)\right) - 2r\arctan\left(\frac{2}{F_r\sqrt{5}}\right)\ln\alpha \tag{5.2}$$

while if r is an odd integer, then

$$\int_0^{\pi/2} \ln\left(\frac{F_r\sqrt{5} + 2\sin x}{F_r\sqrt{5} - 2\sin x}\right) dx = 2\,\mathrm{Cl}_2\left(\arctan\left(\frac{2}{L_r}\right)\right) + 2\,\mathrm{Cl}_2\left(\pi - \arctan\left(\frac{2}{L_r}\right)\right) - 2r\arctan\left(\frac{2}{L_r}\right)\ln\alpha. \tag{5.3}$$

PROOF. Consider r an even integer. Set $q = \beta^r$ in (5.1) and use (1.20), (1.23) and (1.29). Consider r an odd integer. Set $q = -\beta^r$ in (5.1) and use (1.20), (1.23) and (1.30). □

In particular,

$$\int_0^{\pi/2} \ln\left(\frac{1+\sin x}{1-\sin x}\right) dx = 4G, \tag{5.4}$$

which can be compared to other integral representations of G like

$$G = -\int_0^1 \frac{\ln x}{1+x^2}\,dx = \int_0^1 \frac{\tan^{-1} x}{x}\,dx.$$

Differentiating (1.9) gives

$$\int_0^{\pi/2} \frac{\sin x}{\sin^2 x + Q^2}\,dx = \frac{1}{\sqrt{1+Q^2}} \ln\left(\frac{1 - Q + \sqrt{1+Q^2}}{1 + Q - \sqrt{1+Q^2}}\right). \tag{5.5}$$

THEOREM 15. *If r is a non-zero integer, then*

$$\int_0^{\pi/2} \frac{\sin x}{5F_r^2 + L_r^2 \sin^2 x}\,dx = \frac{\sqrt{2}}{2L_r\sqrt{L_{2r}}} \ln\left(\frac{\sqrt{2}\beta^r + \sqrt{L_{2r}}}{\sqrt{2}\alpha^r - \sqrt{L_{2r}}}\right) \tag{5.6}$$

and

$$\int_0^{\pi/2} \frac{\sin x}{L_r^2 + 5F_r^2 \sin^2 x}\,dx = \frac{\sqrt{10}}{10F_r\sqrt{L_{2r}}} \ln\left(\frac{-\sqrt{2}\beta^r + \sqrt{L_{2r}}}{\sqrt{2}\alpha^r - \sqrt{L_{2r}}}\right). \tag{5.7}$$

PROOF. Set $Q = F_r\sqrt{5}/L_r$ and $Q = L_r/(F_r\sqrt{5})$ in (5.5) and simplify making use of $L_n^2 + 5F_n^2 = 2L_{2n}$. □

THEOREM 16. *If $r \geqslant 2$ ($r \geqslant 1$) is an integer, then*

$$\int_0^{\pi/2} \frac{\sin x}{F_{r-1}^2 + F_r^2 \sin^2 x}\,dx = \frac{1}{F_r\sqrt{F_{2r-1}}} \ln\left(\frac{F_{r-2} + \sqrt{F_{2r-1}}}{F_{r+1} - \sqrt{F_{2r-1}}}\right) \tag{5.8}$$

and

$$\int_0^{\pi/2} \frac{\sin x}{L_{r-1}^2 + L_r^2 \sin^2 x}\,dx = \frac{1}{L_r\sqrt{5F_{2r-1}}} \ln\left(\frac{L_{r-2} + \sqrt{5F_{2r-1}}}{L_{r+1} - \sqrt{5F_{2r-1}}}\right). \tag{5.9}$$

PROOF. Set $Q = F_{r-1}/F_r$ and $Q = L_{r-1}/L_r$ in (5.5) and simplify making use of the Catalan identity. □

REMARK. *We mention that identities* (5.8) *and* (5.9) *can be generalized further. For instance, we record that for each $k \geqslant 1$ and each odd $r \geqslant 1$ we have*

$$\int_0^{\pi/2} \frac{\sin x}{F_k^2 + F_{k+r}^2 \sin^2 x}\,dx = \frac{1}{F_{k+r}\sqrt{F_rF_{2k+r}}} \ln\left(\frac{F_{k+r} - F_k + \sqrt{F_rF_{2k+r}}}{F_{k+r} + F_k - \sqrt{F_rF_{2k+r}}}\right), \tag{5.10}$$

which contains (5.8) *and*

$$\int_0^{\pi/2} \frac{\sin x}{F_r^2 + F_{2r}^2 \sin^2 x}\,dx = \frac{1}{F_{2r}\sqrt{F_rF_{3r}}} \ln\left(\frac{F_{2r} - F_r + \sqrt{F_rF_{3r}}}{F_{2r} + F_r - \sqrt{F_rF_{3r}}}\right) \tag{5.11}$$

as special cases.

THEOREM 17. *If r is a positive odd integer, then*

$$\int_0^{\pi/2} \frac{\sin x}{4 + L_r^2 \sin^2 x}\,dx = \frac{r}{\sqrt{5}F_{2r}} \ln\alpha. \tag{5.12}$$

PROOF. Set $Q = 2/L_r$ in (5.5) and simplify. □

COROLLARY 18. *If r is a positive odd integer, then*

$$\int_0^{\pi/2} \frac{\sin^3 x}{(4 + L_r^2 \sin^2 x)^2}\,dx = \frac{1}{10F_{2r}^2}\left(\frac{2rL_{2r}}{\sqrt{5}F_{2r}} \ln\alpha - 1\right). \tag{5.13}$$

PROOF. Differentiate the Fibonacci and Lucas function forms of (5.12) and take the real part, using (1.18). □

REMARK. *Noting that*

$$\int_0^{\pi/2} \frac{\sin x}{\sin^2 x + Q^2}\,dx = \frac{1}{\sqrt{1 + Q^2}} \ln\left(\frac{1 + \sqrt{1 + Q^2}}{Q}\right).$$

it is clear the results presented in this section can be stated in a slightly different form.

6. Results associated with (1.10)

We can write (1.10) as

$$\int_0^\pi x\arctan(Q\sin x)\,dx = \pi\,\mathrm{Li}_2(q) - \pi\,\mathrm{Li}_2(-q), \tag{6.1}$$

where

$$Q = Q(q) = \frac{2q}{1-q^2}; \tag{6.2}$$

so that

$$\frac{dQ}{dq} = \frac{2(1+q^2)}{(1-q^2)^2} = \frac{1}{(1-q)^2} + \frac{1}{(1+q)^2}. \tag{6.3}$$

Differentiating (6.1) with respect to q, we have

$$\int_0^\pi \frac{x\sin x}{1+Q^2\sin^2 x}\,dx = \frac{\pi\ln\left(\left|\dfrac{1+q}{1-q}\right|\right)}{q\dfrac{dQ}{dq}}; \tag{6.4}$$

that is

$$\int_0^\pi \frac{x\sin x}{1+\left(\dfrac{2q}{1-q^2}\right)^2\sin^2 x}\,dx = \frac{\pi}{2}\frac{\ln\left(\left|\dfrac{1+q}{1-q}\right|\right)}{\dfrac{q}{1-q^2}\dfrac{1+q^2}{1-q^2}}. \tag{6.5}$$

We now proceed to derive from (6.5) a couple of identities involving Fibonacci and Lucas numbers.

Theorem 19. *If r is a non-zero integer, then*

$$\int_0^\pi \frac{x\sin x}{L_r^2+4\sin^2 x}\,dx = \frac{\pi}{2F_r\sqrt5}\ln\left(\frac{F_r\sqrt5+2}{L_r}\right), \quad r \text{ odd}; \tag{6.6}$$

$$\int_0^\pi \frac{x\sin x}{L_r^2-4\sin^2 x}\,dx = \frac{\pi}{2L_r}\ln\left(\frac{F_r\sqrt5}{L_r-2}\right), \quad r \text{ even}. \tag{6.7}$$

Proof. Set $q=\beta^r$ in (6.5) and use (1.20). □

Corollary 20. *If r is a non-zero integer, then*

$$\int_0^\pi \frac{x\sin x}{\left(L_r^2+4\sin^2 x\right)^2}\,dx = \frac{\pi}{20\sqrt5F_r^3}\ln\left(\frac{F_r\sqrt5+2}{L_r}\right) + \frac{\pi}{10F_{2r}^2}, \quad r \text{ odd}; \tag{6.8}$$

$$\int_0^\pi \frac{x\sin x}{\left(L_r^2-4\cos^2 x\right)^2}\,dx = \frac{\pi}{4L_r^3}\ln\left(\frac{F_r\sqrt5}{L_r-2}\right) + \frac{\pi}{10F_{2r}^2}, \quad r \text{ even}. \tag{6.9}$$

Proof. Differentiate the Fibonacci and Lucas function forms of (6.6) and (6.7) and take the real part, using (1.18). □

THEOREM 21. *If r is a non-negative integer, then*

$$\int_0^\pi \frac{x\sin x}{4+5F_{2r}^2\sin^2 x}\,dx = \frac{2\pi r\sqrt{5}}{5F_{4r}}\ln\alpha. \tag{6.10}$$

PROOF. Set $q = F_r\sqrt{5}/L_r$ in (6.5). □

COROLLARY 22. *If r is a non-negative integer, then*

$$\int_0^\pi \frac{x\sin^3 x}{\left(4+5F_{2r}^2\sin^2 x\right)^2}\,dx = -\frac{1}{10}\frac{\pi}{F_{4r}^2} + \frac{2\pi\sqrt{5}}{25}\frac{L_{4r}}{F_{4r}^3}r\ln\alpha. \tag{6.11}$$

PROOF. Differentiate the Fibonacci-Lucas function form of (6.10) and take the real part, using (1.18). □

Next write (1.10) as

$$\int_0^\pi x\arctan(Q\sin x)\,dx = \pi\operatorname{Li}_2\left(\frac{\sqrt{1+Q^2}-1}{Q}\right) - \pi\operatorname{Li}_2\left(\frac{-\sqrt{1+Q^2}+1}{Q}\right), \quad Q\in\mathbb{R}, \tag{6.12}$$

which by writing iQ for Q also implies

$$\begin{aligned}\int_0^\pi x\ln\left(\frac{1+Q\sin x}{1-Q\sin x}\right)dx \\ = 2i\pi\operatorname{Li}_2\left(i\frac{\left(\sqrt{1-Q^2}-1\right)}{Q}\right) - 2i\pi\operatorname{Li}_2\left(-i\frac{\left(\sqrt{1-Q^2}-1\right)}{Q}\right), \quad Q^2<1,\end{aligned} \tag{6.13}$$

and which upon differentiation gives

$$\int_0^\pi \frac{x\sin x}{1+Q^2\sin^2 x}\,dx = \frac{\pi}{Q\sqrt{1+Q^2}}\ln\left(Q+\sqrt{1+Q^2}\right), \quad Q\in\mathbb{R}. \tag{6.14}$$

REMARK. *By setting $Q = 2/L_r$ and $Q = 2/F_r\sqrt{5}$, in turn, in* (6.13)*, similar results to those in Theorem 14 can be derived.*

THEOREM 23. *If r is a non-zero integer, then*

$$\int_0^\pi \frac{x\sin x}{2L_{2r}-L_r^2\cos^2 x}\,dx = \frac{\pi\sqrt{2}}{2L_r\sqrt{L_{2r}}}\ln\left(\frac{\beta^r\sqrt{2}+\sqrt{L_{2r}}}{\alpha^r\sqrt{2}-\sqrt{L_{2r}}}\right), \tag{6.15}$$

$$\int_0^\pi \frac{x\sin x}{2L_{2r}-5F_r^2\cos^2 x}\,dx = \frac{\pi\sqrt{5}\sqrt{2}}{10F_r\sqrt{L_{2r}}}\ln\left(\frac{-\beta^r\sqrt{2}+\sqrt{L_{2r}}}{\alpha^r\sqrt{2}-\sqrt{L_{2r}}}\right). \tag{6.16}$$

PROOF. Set $Q = L_r/(F_r\sqrt{5})$ in (6.14) to obtain (6.15) and $Q = F_r\sqrt{5}/L_r$ to obtain (6.16). □

Writing iQ for Q in (6.14), we have

$$\int_0^\pi \frac{x\sin x}{1-Q^2\sin^2 x}\,dx = \frac{\pi}{Q\sqrt{1-Q^2}}\arctan\left(\frac{Q}{\sqrt{1-Q^2}}\right), \quad Q^2<1. \tag{6.17}$$

THEOREM 24. *If r is a non-zero even integer, then*

$$\int_0^\pi \frac{x\sin x}{L_r^2-4\sin^2 x}\,dx = \frac{\pi}{2}\frac{1}{F_r\sqrt{5}}\arctan\left(\frac{2}{F_r\sqrt{5}}\right), \tag{6.18}$$

while if r is an odd integer, then

$$\int_0^\pi \frac{x\sin x}{5F_r^2-4\sin^2 x}\,dx = \frac{\pi}{2}\frac{1}{L_r}\arctan\left(\frac{2}{L_r}\right). \tag{6.19}$$

PROOF. Set $Q=2/L_r$ and $Q=2/F_r\sqrt{5}$, in turn, in (6.17) . □

In particular,

$$\int_0^\pi \frac{x\sin x}{5-4\sin^2 x}\,dx = \frac{\pi}{2}\arctan 2. \tag{6.20}$$

COROLLARY 25. *If r is a non-zero even integer, then*

$$\int_0^\pi \frac{x\sin x}{\left(L_r^2-4\sin^2 x\right)^2}\,dx = \frac{\pi}{20\sqrt{5}F_r^3}\arctan\left(\frac{2}{F_r\sqrt{5}}\right)+\frac{\pi}{10F_{2r}^2}, \tag{6.21}$$

while if r is an odd integer, then

$$\int_0^\pi \frac{x\sin x}{\left(5F_r^2-4\sin^2 x\right)^2}\,dx = \frac{\pi}{4L_r^3}\arctan\left(\frac{2}{L_r}\right)+\frac{\pi}{10F_{2r}^2}. \tag{6.22}$$

PROOF. Differentiate the Fibonacci-Lucas function forms of the identities in Theorem 24. □

REMARK. *More identities can be derived through the following identities, valid for $Q^2<1$, obtained from the addition and subtraction of* (6.14) *and* (6.17)*:*

(6.23)

$$\int_0^\pi \frac{x\sin x}{1-Q^4\sin^4 x}\,dx = \frac{\pi}{2Q\sqrt{1-Q^2}}\arctan\left(\frac{Q}{\sqrt{1-Q^2}}\right)+\frac{\pi}{2Q\sqrt{1+Q^2}}\ln\left(Q+\sqrt{1+Q^2}\right),$$

(6.24)

$$\int_0^\pi \frac{x\sin^3 x}{1-Q^4\sin^4 x}\,dx = \frac{\pi}{2Q^3\sqrt{1-Q^2}}\arctan\left(\frac{Q}{\sqrt{1-Q^2}}\right)-\frac{\pi}{2Q^3\sqrt{1+Q^2}}\ln\left(Q+\sqrt{1+Q^2}\right).$$

REMARK. *Replacing Q with $1/Q$ in* (5.5) *yields*

$$\int_0^{\pi/2} \frac{\sin x}{1+Q^2\sin^2 x}\,dx = \frac{1}{Q\sqrt{1+Q^2}}\ln\left(Q+\sqrt{1+Q^2}\right), \tag{6.25}$$

which upon comparison with (6.14) *proves the following relation valid for all Q*

$$\int_0^{\pi/2} \frac{\sin x}{1+Q^2\sin^2 x}\,dx = \frac{1}{\pi}\int_0^\pi \frac{x\sin x}{1+Q^2\sin^2 x}\,dx. \tag{6.26}$$

In fact, (6.26) *implies that*

$$\int_0^{\pi/2} \frac{\sin^{2m-1} x}{\left(1+Q^2\sin^2 x\right)^m}\,dx = \frac{1}{\pi}\int_0^\pi \frac{x\sin^{2m-1} x}{\left(1+Q^2\sin^2 x\right)^m}\,dx, \qquad m\in\mathbb{N},\ Q\in\mathbb{C}. \tag{6.27}$$

7. Results associated with (1.11)

THEOREM 26. *If r is a non-zero integer, then*

$$\int_0^{\pi/2} \frac{x^2}{L_r + 2\cos(2x)}\,dx = \frac{1}{\sqrt{5}F_r}\left(\frac{\pi^3}{24} + \frac{\pi}{2}\operatorname{Li}_2(\beta^r)\right), \quad r \text{ even}; \tag{7.1}$$

$$\int_0^{\pi/2} \frac{x^2}{\sqrt{5}F_r - 2\cos(2x)}\,dx = \frac{1}{L_r}\left(\frac{\pi^3}{24} + \frac{\pi}{2}\operatorname{Li}_2(\beta^r)\right), \quad r \text{ odd}. \tag{7.2}$$

In particular,

$$\int_0^{\pi/2} \frac{x^2}{3 + 2\cos(2x)}\,dx = \frac{1}{\sqrt{5}}\left(\frac{3\pi^3}{40} - \frac{\pi}{2}\ln^2\alpha\right) \tag{7.3}$$

and

$$\int_0^{\pi/2} \frac{x^2}{\sqrt{5} - 2\cos(2x)}\,dx = \frac{\pi^3}{120} + \frac{\pi}{4}\ln^2\alpha. \tag{7.4}$$

PROOF. Set $q = -\beta^r$ in (1.11) and use (1.20). The special cases follow from the evaluations

$$\operatorname{Li}_2(\beta^2) = \frac{\pi^2}{15} - \ln^2\alpha \quad \text{and} \quad \operatorname{Li}_2(\beta) = -\frac{\pi^2}{15} + \frac{\ln^2\alpha}{2}. \tag{7.5}$$

□

THEOREM 27. *If r is a non-zero integer, then*

$$\int_0^{\pi/2} \frac{x^2}{(L_r + 2\cos(2x))^2}\,dx = \frac{L_r}{(\sqrt{5}F_r)^3}\left(\frac{\pi^3}{24} + \frac{\pi}{2}\operatorname{Li}_2(\beta^r)\right) - \frac{\pi}{2}\frac{\ln(1-\beta^r)}{5F_r^2}, \quad r \text{ even}; \tag{7.6}$$

$$\int_0^{\pi/2} \frac{x^2}{(\sqrt{5}F_r - 2\cos(2x))^2}\,dx = \frac{\sqrt{5}F_r}{L_r^3}\left(\frac{\pi^3}{24} + \frac{\pi}{2}\operatorname{Li}_2(\beta^r)\right) - \frac{\pi}{2}\frac{\ln(1-\beta^r)}{L_r^2}, \quad r \text{ odd}. \tag{7.7}$$

In particular,

$$\int_0^{\pi/2} \frac{x^2}{(3 + 2\cos(2x))^2}\,dx = \frac{3}{5\sqrt{5}}\left(\frac{3\pi^3}{40} - \frac{\pi}{2}\ln^2\alpha\right) + \frac{\pi}{10}\ln\alpha \tag{7.8}$$

and

$$\int_0^{\pi/2} \frac{x^2}{(\sqrt{5} - 2\cos(2x))^2}\,dx = \sqrt{5}\left(\frac{\pi^3}{120} + \frac{\pi}{4}\ln^2\alpha\right) - \frac{\pi}{2}\ln\alpha. \tag{7.9}$$

PROOF. Differentiate the Fibonacci-Lucas function forms of (7.1) and (7.2), and take the real part, using (1.18). □

THEOREM 28. *If $r \geqslant 2$ is an even integer, then*

$$\int_0^{\pi/2} \frac{x^2}{L_{2r} + \sqrt{5}F_{2r}\cos(2x)}\,dx = \frac{\pi^3}{48} + \frac{\pi}{4}\operatorname{Li}_2\left(\frac{\sqrt{5}F_r}{L_r}\right). \tag{7.10}$$

In particular,

$$\int_0^{\pi/2} \frac{x^2}{7+3\sqrt{5}\cos(2x)}\,dx = \frac{\pi^3}{48} + \frac{\pi}{4}\operatorname{Li}_2\left(\frac{\sqrt{5}}{3}\right). \tag{7.11}$$

PROOF. Set $q = -\sqrt{5}F_r/L_r$ in (1.11) and keep in mind that $q < -1$ for r being odd and $-1 < q < 0$ for r being even. □

THEOREM 29. *If $r \geqslant 2$ is an even integer, then*

$$\int_0^{\pi/2} \frac{x^2(\sqrt{5}F_{2r} + L_{2r}\cos(2x))}{(L_{2r} + \sqrt{5}F_{2r}\cos(2x))^2}\,dx = \frac{\pi}{2\sqrt{5}F_{2r}} \ln\left(1 - \frac{\sqrt{5}F_r}{L_r}\right). \tag{7.12}$$

In particular,

$$\int_0^{\pi/2} \frac{x^2(3\sqrt{5} + 7\cos(2x))}{(7 + 3\sqrt{5}\cos(2x))^2}\,dx = \frac{\pi}{6\sqrt{5}} \ln\left(\frac{2}{3\alpha^2}\right). \tag{7.13}$$

PROOF. Differentiate the Fibonacci-Lucas function form of (7.10) and take the real part, using (1.18). □

THEOREM 30. *If $r \geqslant 2$ is an integer, then*

$$\int_0^{\pi/2} \frac{x^2}{L_r^2 + 4 + 4L_r\cos(2x)}\,dx = \frac{1}{L_r^2 - 4}\left(\frac{\pi^3}{24} + \frac{\pi}{2}\operatorname{Li}_2\left(\frac{2}{L_r}\right)\right). \tag{7.14}$$

In particular,

$$\int_0^{\pi/2} \frac{x^2}{5 + 4\cos(2x)}\,dx = \frac{\pi^3}{36} - \frac{\pi}{12}\ln^2 2. \tag{7.15}$$

PROOF. Set $q = -2/L_r$ in (1.11). The special case follows from the evaluation

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2.$$

□

THEOREM 31. *If $r \geqslant 2$ is an integer, then*

(7.16)

$$\int_0^{\pi/2} \frac{x^2(L_r + 2\cos(2x))}{(L_r^2 + 4 + 4L_r\cos(2x))^2}\,dx = \frac{L_r}{(L_r^2 - 4)^2}\left(\frac{\pi^3}{24} + \frac{\pi}{2}\operatorname{Li}_2\left(\frac{2}{L_r}\right)\right) - \frac{\pi}{4}\frac{1}{L_r(L_r^2 - 4)}\ln\left(1 - \frac{2}{L_r}\right).$$

In particular,

$$\int_0^{\pi/2} \frac{x^2(2 + \cos(2x))}{(5 + 4\cos(2x))^2}\,dx = \frac{\pi^3}{54} - \frac{\pi}{18}\ln^2 2 + \frac{\pi}{24}\ln 2. \tag{7.17}$$

PROOF. Differentiate the Fibonacci-Lucas function form of (7.14) and take the real part, using (1.18). □

8. Results associated with (1.12)

THEOREM 32. *If r is a positive integer, then*

$$\frac{1}{F_{2r}\sqrt{5}}\left(\frac{\pi^3}{3}+\pi\,\mathrm{Li}_2(\beta^{2r})\right)=\begin{cases}\int_0^\pi \frac{x^2}{L_r^2-4\cos^2 x}\,dx, & \text{if } r \text{ is even;}\\ \int_0^\pi \frac{x^2}{L_r^2+4\sin^2 x}\,dx, & \text{if } r \text{ is odd.}\end{cases} \tag{8.1}$$

In particular,

$$\int_0^\pi \frac{x^2}{1+4\sin^2 x}\,dx=\frac{2\pi^3}{5\sqrt{5}}-\frac{\pi\ln^2\alpha}{\sqrt{5}}. \tag{8.2}$$

PROOF. Set $q=\beta^{2r}$ in (1.12) and use (1.20). Note the use of the Fibonacci-Lucas fundamental identity $L_r^2-5F_r^2=(-1)^r 4$. □

COROLLARY 33. *If r is a positive integer, then*

$$\int_0^\pi \frac{x^2}{(L_r^2-4\cos^2 x)^2}\,dx=\frac{\pi^3\sqrt{5}}{75}\frac{L_{2r}}{F_{2r}^3}+\frac{\pi\sqrt{5}}{25}\frac{L_{2r}}{F_{2r}^3}\mathrm{Li}_2(\beta^{2r})-\frac{\pi}{5F_{2r}^2}\ln\left(\beta^r F_r\sqrt{5}\right),\ r \text{ even}, \tag{8.3}$$

$$\int_0^\pi \frac{x^2}{\left(L_r^2+4\sin^2 x\right)^2}\,dx=\frac{\pi^3\sqrt{5}}{75}\frac{L_{2r}}{F_{2r}^3}+\frac{\pi\sqrt{5}}{25}\frac{L_{2r}}{F_{2r}^3}\mathrm{Li}_2(\beta^{2r})-\frac{\pi}{5F_{2r}^2}\ln\left(-\beta^r L_r\right),\ r \text{ odd}. \tag{8.4}$$

In particular,

$$\int_0^\pi \frac{x^2}{\left(1+4\sin^2 x\right)^2}\,dx=\frac{6\pi^3\sqrt{5}}{125}-\frac{3\pi\sqrt{5}}{25}\ln^2\alpha+\frac{\pi}{5}\ln\alpha. \tag{8.5}$$

PROOF. Differentiate the Fibonacci-Lucas function forms of (8.1) with respect to r and use (1.18). □

Identity (1.12) can also be written as

$$\int_0^\pi \frac{x^2}{1-Q\cos^2 x}\,dx=\frac{1}{\sqrt{1-Q}}\left(\frac{\pi^3}{3}+\pi\,\mathrm{Li}_2\left(\frac{2-Q-2\sqrt{1-Q}}{Q}\right)\right), Q<1; \tag{8.6}$$

from which we can obtain more results.

THEOREM 34. *If r is a non-zero integer, then*

$$\int_0^\pi \frac{x^2}{L_r^2-4(-1)^r\cos^2 x}\,dx=\frac{1}{F_{2r}\sqrt{5}}\left(\frac{\pi^3}{3}+\pi\,\mathrm{Li}_2\left((-1)^r\beta^{2r}\right)\right). \tag{8.7}$$

PROOF. Setting $Q=\sin^2 z$ in (8.6) gives

$$\int_0^\pi \frac{x^2}{1-\sin^2 z\cos^2 x}\,dx=\frac{1}{\cos z}\left(\frac{\pi^3}{3}+\pi\,\mathrm{Li}_2\left(\frac{(1-\cos z)^2}{\sin^2 z}\right)\right), \tag{8.8}$$

from which (8.7) follows upon use of (1.15). □

COROLLARY 35. *If r is a non-zero integer, then*

$$\int_0^\pi \frac{x^2}{\left(L_r^2 - 4(-1)^r \cos^2 x\right)^2}\,dx = \frac{\pi^3\sqrt{5}}{75}\frac{L_{2r}}{F_{2r}^3} + \frac{\pi\sqrt{5}}{25}\frac{L_{2r}}{F_{2r}^3}\operatorname{Li}_2\left((-1)^r\beta^{2r}\right) - \frac{\pi}{5F_{2r}^2}\ln\left(1-(-1)^r\beta^{2r}\right). \tag{8.9}$$

PROOF. Differentiate the Fibonacci-Lucas function form of (8.7), using (1.18). □

THEOREM 36. *If r is a non-zero integer, then*

$$\int_0^\pi \frac{x^2\cos^2 x}{\left(L_r^2 - 4(-1)^r \cos^2 x\right)^2}\,dx = \left(-\frac{1}{10}\frac{\pi}{F_rF_{2r}\beta^r} + \frac{\pi\sqrt{5}}{20}\frac{(-1)^r}{F_{2r}}\right)\ln\left(1-(-1)^r\beta^{2r}\right) + \frac{\pi^3\sqrt{5}}{150F_{2r}F_r^2} + \frac{\pi\sqrt{5}}{50F_{2r}F_r^2}\operatorname{Li}_2\left((-1)^r\beta^{2r}\right). \tag{8.10}$$

PROOF. Differentiate (8.8) with respect to z and use (1.15). □

REMARK. *Equivalent/similar results to* (8.10) *can be obtained directly by substituting $q = \beta^{2r}$ in the following identity obtained by differentiating* (1.12) *with respect to q:*

$$\int_0^\pi \frac{x^2\cos^2 x}{(1-Q\cos^2 x)^2}\,dx = -\frac{\pi}{4}\frac{(1+q)^4}{(1-q)^2}\frac{\ln(1-q)}{q} + \frac{1}{2}\left(\frac{1+q}{1-q}\right)^3\left(\frac{\pi^3}{3} + \pi\operatorname{Li}_2(q)\right).$$

9. Results associated with (1.13)

LEMMA 9. *Let $q < 1$ and let*

$$Q = \frac{2q}{1+q^2}. \tag{9.1}$$

Then

$$\int_0^\pi \frac{x^2\,dx}{1 \pm Q\cos(2x)} = \frac{1+q^2}{1-q^2}\left(\frac{\pi^3}{3} + \pi\operatorname{Li}_2(\pm q)\right). \tag{9.2}$$

THEOREM 37. *If r is an integer, then*

$$\frac{\pi^3}{3} + \pi\operatorname{Li}_2(\pm\beta^r) = \begin{cases} F_r\sqrt{5}\int_0^\pi \dfrac{x^2\,dx}{L_r \mp 2\cos(2x)}, & \text{if } r \text{ is even;} \\ L_r\int_0^\pi \dfrac{x^2\,dx}{F_r\sqrt{5} \pm 2\cos(2x)}, & \text{if } r \text{ is odd.} \end{cases} \tag{9.3}$$

PROOF. Set $q = \beta^r$ in (9.2) and use (1.20). □

In particular,

$$\int_0^\pi \frac{x^2\,dx}{\sqrt{5}+2\cos(2x)} = \frac{4\pi^3}{15} + \frac{\pi}{2}\ln^2\alpha, \tag{9.4}$$

$$\int_0^\pi \frac{x^2\,dx}{\sqrt{5}-2\cos(2x)} = \frac{13\pi^3}{30} - \pi\ln^2\alpha, \tag{9.5}$$

$$\int_0^\pi \frac{x^2\,dx}{3-2\cos(2x)} = \frac{1}{\sqrt{5}}\left(\frac{2\pi^3}{5} - \pi\ln^2\alpha\right); \tag{9.6}$$

where we used (7.5) and also

$$\operatorname{Li}_2(-\beta) = \frac{\pi^2}{10} - \ln^2\alpha, \quad \operatorname{Li}_2(\beta) = -\frac{\pi^2}{15} + \frac{1}{2}\ln^2\alpha. \tag{9.7}$$

Theorem 38. *If r is a positive even integer, then*

$$\int_0^\pi \frac{x^2\,dx}{\left(L_r \mp 2\cos(2x)\right)^2} = -\frac{\pi}{5F_r^2}\ln\left(1\mp\beta^r\right) + \frac{L_r}{5F_r^3\sqrt{5}}\left(\frac{\pi^3}{3} + \pi\operatorname{Li}_2\left(\pm\beta^r\right)\right), \tag{9.8}$$

while if r is a positive odd integer, then

$$\int_0^\pi \frac{x^2\,dx}{\left(F_r\sqrt{5} \pm 2\cos(2x)\right)^2} = -\frac{\pi}{L_r^2}\ln\left(1\mp\beta^r\right) + \frac{F_r\sqrt{5}}{L_r^3}\left(\frac{\pi^3}{3} + \pi\operatorname{Li}_2\left(\pm\beta^r\right)\right). \tag{9.9}$$

Proof. Differentiate the Fibonacci-Lucas function forms of (9.3). □

In particular,

$$\int_0^\pi \frac{x^2\,dx}{\left(3-2\cos(2x)\right)^2} = \frac{\pi}{5}\ln\alpha + \frac{6\pi^3}{25\sqrt{5}} - \frac{3\pi}{5\sqrt{5}}\ln^2\alpha, \tag{9.10}$$

$$\int_0^\pi \frac{x^2\,dx}{\left(\sqrt{5}+2\cos(2x)\right)^2} = -\pi\ln\alpha + \frac{4\pi^3}{3\sqrt{5}} + \frac{\pi\sqrt{5}}{2}\ln^2\alpha, \tag{9.11}$$

$$\int_0^\pi \frac{x^2\,dx}{\left(\sqrt{5}-2\cos(2x)\right)^2} = 2\pi\ln\alpha + \frac{13\pi^3}{6\sqrt{5}} - \pi\sqrt{5}\ln^2\alpha. \tag{9.12}$$

Corollary 39. *If r is an even integer, then*

(9.13)
$$\int_0^\pi \frac{\left(L_r^2+4\cos^2(2x)\right)x^2}{\left(L_r^2-4\cos^2(2x)\right)^2}\,dx = -\frac{\pi}{10F_r^2}\ln\left(\beta^r F_r\sqrt{5}\right) + \frac{L_r}{20F_r^3\sqrt{5}}\left(\frac{4\pi^3}{3} + \pi\operatorname{Li}_2\left(\beta^{2r}\right)\right),$$

(9.14)
$$\int_0^\pi \frac{x^2\cos(2x)}{\left(L_r^2-4\cos^2(2x)\right)^2}\,dx = \frac{\pi}{40F_r^2L_r}\ln\left(\frac{1+\beta^r}{1-\beta^r}\right) + \frac{\pi}{40F_r^3\sqrt{5}}\left(\operatorname{Li}_2\left(\beta^r\right) - \operatorname{Li}_2\left(-\beta^r\right)\right)$$

while if r is an odd integer, then

(9.15)
$$\int_0^\pi \frac{\left(5F_r^2+4\cos^2(2x)\right)x^2}{\left(5F_r^2-4\cos^2(2x)\right)^2}\,dx = -\frac{\pi}{2L_r^2}\ln\left(-\beta^r L_r\right)+\frac{F_r\sqrt{5}}{4L_r^3}\left(\frac{4\pi^3}{3}+\pi\,\mathrm{Li}_2\left(\beta^{2r}\right)\right),$$

(9.16)
$$\int_0^\pi \frac{x^2\cos(2x)\,dx}{\left(5F_r^2-4\cos^2(2x)\right)^2}=\frac{\pi}{8L_r^2F_r\sqrt{5}}\ln\left(\frac{1-\beta^r}{1+\beta^r}\right)-\frac{\pi}{8L_r^3}\left(\mathrm{Li}_2\left(\beta^r\right)-\mathrm{Li}_2\left(-\beta^r\right)\right).$$

PROOF. Identities (9.13) and (9.14) are obtained from the respective addition and subtraction of the two identities contained in (9.8) while (9.15) and (9.16) follow from (9.9). □

In particular,

$$\int_0^\pi \frac{\left(5+4\cos^2(2x)\right)x^2}{\left(5-4\cos^2(2x)\right)^2}\,dx=\frac{\pi}{2}\ln\alpha+\frac{7\pi^3}{4\sqrt{5}}-\frac{\pi\sqrt{5}}{4}\ln^2\alpha, \tag{9.17}$$

$$\int_0^\pi \frac{x^2\cos(2x)}{\left(5-4\cos^2(2x)\right)^2}\,dx=\frac{3\pi}{8\sqrt{5}}\ln\alpha+\frac{\pi^3}{48}-\frac{3\pi}{16}\ln^2\alpha. \tag{9.18}$$

LEMMA 10. *If* $0<q<1$, *then*

$$\int_0^\pi \frac{x^2\,dx}{1-Q^2\cos^2(2x)}=\frac{1+q^2}{1-q^2}\left(\frac{\pi^3}{3}+\frac{\pi}{4}\,\mathrm{Li}_2(q^2)\right), \tag{9.19}$$

$$\int_0^\pi \frac{x^2\cos(2x)\,dx}{1-Q^2\cos^2(2x)}=\frac{\pi}{2Q}\frac{1+q^2}{1-q^2}\left(\mathrm{Li}_2(q)-\mathrm{Li}_2(-q)\right), \tag{9.20}$$

where Q *is as given in* (9.1).

PROOF. Immediate consequence of the identities in Lemma 9. We also used

$$\mathrm{Li}_2(y)+\mathrm{Li}_2(-y)=\frac{1}{2}\,\mathrm{Li}_2(y^2).$$

□

THEOREM 40. *If r is a positive integer, then*

$$\frac{1}{F_{2r}\sqrt{5}}\left(\frac{\pi^3}{3}+\frac{\pi}{4}\,\mathrm{Li}_2\left(\beta^{2r}\right)\right)=\begin{cases}\int_0^\pi \dfrac{x^2\,dx}{L_r^2-4\cos^2(2x)}, & \text{if } r \text{ is even;}\\ \int_0^\pi \dfrac{x^2\,dx}{L_r^2+4\sin^2(2x)}, & \text{if } r \text{ is odd;}\end{cases} \tag{9.21}$$

and

$$\frac{\pi}{4}\left(\mathrm{Li}_2\left(\beta^r\right)-\mathrm{Li}_2\left(-\beta^r\right)\right)=\begin{cases}F_r\sqrt{5}\int_0^\pi \dfrac{x^2\cos(2x)\,dx}{5F_r^2+4\sin^2(2x)}, & \text{if } r \text{ is even;}\\ -L_r\int_0^\pi \dfrac{x^2\cos(2x)\,dx}{5F_r^2-4\cos^2(2x)}, & \text{if } r \text{ is odd.}\end{cases} \tag{9.22}$$

PROOF. Set $q=\beta^r$ in (9.19) and (9.20). □

In particular,

$$\int_0^\pi \frac{x^2\,dx}{1+4\sin^2(2x)} = \frac{1}{\sqrt5}\left(\frac{7\pi^3}{20} - \frac{\pi}{4}\ln^2\alpha\right), \tag{9.23}$$

$$\int_0^\pi \frac{x^2\cos(2x)\,dx}{5-4\cos^2(2x)} = \frac{\pi^3}{24} - \frac{3\pi}{8}\ln^2\alpha. \tag{9.24}$$

Corollary 41. *If r is a positive even integer, then*

$$\int_0^\pi \frac{x^2\cos^2 x}{L_r^2 - 4\cos^2(2x)}\,dx = \frac{1}{F_{2r}\sqrt5}\left(\frac{\pi^3}{6} + \frac{\pi}{8}\operatorname{Li}_2\left(\beta^{2r}\right)\right) + \frac{\pi}{8F_r\sqrt5}\left(\operatorname{Li}_2\left(\beta^r\right) - \operatorname{Li}_2\left(-\beta^r\right)\right), \tag{9.25}$$

$$\int_0^\pi \frac{x^2\sin^2 x}{L_r^2 - 4\cos^2(2x)}\,dx = \frac{1}{F_{2r}\sqrt5}\left(\frac{\pi^3}{6} + \frac{\pi}{8}\operatorname{Li}_2\left(\beta^{2r}\right)\right) - \frac{\pi}{8F_r\sqrt5}\left(\operatorname{Li}_2\left(\beta^r\right) - \operatorname{Li}_2\left(-\beta^r\right)\right), \tag{9.26}$$

while if r is a positive odd number, then

$$\int_0^\pi \frac{x^2\sin^2 x}{L_r^2 + 4\sin^2(2x)}\,dx = \frac{1}{F_{2r}\sqrt5}\left(\frac{\pi^3}{6} + \frac{\pi}{8}\operatorname{Li}_2\left(\beta^{2r}\right)\right) + \frac{\pi}{8L_r}\left(\operatorname{Li}_2\left(\beta^r\right) - \operatorname{Li}_2\left(-\beta^r\right)\right), \tag{9.27}$$

$$\int_0^\pi \frac{x^2\cos^2 x}{L_r^2 + 4\sin^2(2x)}\,dx = \frac{1}{F_{2r}\sqrt5}\left(\frac{\pi^3}{6} + \frac{\pi}{8}\operatorname{Li}_2\left(\beta^{2r}\right)\right) - \frac{\pi}{8L_r}\left(\operatorname{Li}_2\left(\beta^r\right) - \operatorname{Li}_2\left(-\beta^r\right)\right). \tag{9.28}$$

Proof. Addition and subtraction of corresponding identities in (9.21) and (9.22). □

In particular,

$$\int_0^\pi \frac{x^2\sin^2 x}{1+4\sin^2(2x)}\,dx = \left(-\frac{\sqrt5}{40} + \frac{3}{16}\right)\pi\ln^2\alpha + \left(\frac{7\sqrt5}{200} - \frac{1}{48}\right)\pi^3, \tag{9.29}$$

$$\int_0^\pi \frac{x^2\cos^2 x}{1+4\sin^2(2x)}\,dx = -\left(\frac{\sqrt5}{40} + \frac{3}{16}\right)\pi\ln^2\alpha + \left(\frac{7\sqrt5}{200} + \frac{1}{48}\right)\pi^3. \tag{9.30}$$

Theorem 42. *If r is a positive integer, then*

$$\begin{aligned}&\frac{L_{2r}}{5F_{2r}^3\sqrt5}\left(\frac{\pi^3}{3} + \frac{\pi}{4}\operatorname{Li}_2\left(\beta^{2r}\right)\right) - \frac{\pi}{20F_{2r}^2}\ln\left(1-\beta^{2r}\right)\\ &= \begin{cases}\int_0^\pi \dfrac{x^2\,dx}{\left(L_r^2 - 4\cos^2(2x)\right)^2}, & \text{if } r \text{ is even;}\\ \int_0^\pi \dfrac{x^2\,dx}{\left(L_r^2 + 4\sin^2(2x)\right)^2}, & \text{if } r \text{ is odd;}\end{cases}\end{aligned} \tag{9.31}$$

and

(9.32)
$$\frac{\pi}{8F_{2r}\sqrt{5}}\ln\left(\frac{1+\beta^r}{1-\beta^r}\right)$$
$$=\begin{cases}-\dfrac{\pi}{40F_r^2}\left(\operatorname{Li}_2\left(\beta^r\right)-\operatorname{Li}_2\left(-\beta^r\right)\right)+F_r\sqrt{5}\int_0^\pi\dfrac{x^2\cos(2x)\,dx}{\left(5F_r^2+4\sin^2(2x)\right)^2}, & r \text{ even};\\ -\dfrac{\pi}{8L_r^2}\left(\operatorname{Li}_2\left(\beta^r\right)-\operatorname{Li}_2\left(-\beta^r\right)\right)-L_r\int_0^\pi\dfrac{x^2\cos(2x)\,dx}{\left(5F_r^2-4\cos^2(2x)\right)^2}, & r \text{ odd}.\end{cases}$$

PROOF. Differentiate the Fibonacci-Lucas function form of (9.21) to obtain (9.31); differentiate the Fibonacci-Lucas function form of (9.22) to obtain (9.32). □

In particular,

$$\int_0^\pi\frac{x^2\,dx}{\left(1+4\sin^2(2x)\right)^2}=\frac{21}{100}\frac{\pi^3}{\sqrt{5}}-\frac{3\pi}{20\sqrt{5}}\ln^2\alpha+\frac{\pi}{20}\ln\alpha, \tag{9.33}$$

$$\int_0^\pi\frac{x^2\cos(2x)\,dx}{\left(5-4\cos^2(2x)\right)^2}=\frac{3\pi}{8\sqrt{5}}\ln\alpha+\frac{\pi^3}{48}-\frac{3\pi}{16}\ln^2\alpha. \tag{9.34}$$

LEMMA 11. *Let* $0<q<1$. *Let*

$$R=\frac{2q}{1-q^2}.$$

Then

$$\int_0^\pi\frac{x^2\,dx}{1+R^2\cos^2(2x)}=\frac{1-q^2}{1+q^2}\left(\frac{\pi^3}{3}+\frac{\pi}{4}\operatorname{Li}_2\left(-q^2\right)\right) \tag{9.35}$$

and

(9.36)
$$\int_0^\pi\frac{x^2\cos(2x)\,dx}{1+R^2\cos^2(2x)}$$
$$=\frac{\pi}{R}\frac{1-q^2}{1+q^2}\left(\arctan q\ln q+\frac{1}{2}\operatorname{Cl}_2\left(2\arctan q\right)+\frac{1}{2}\operatorname{Cl}_2\left(\pi-2\arctan q\right)\right).$$

PROOF. Write iq for q in (1.13) and take real and imaginary parts to obtain

$$\int_0^\pi\frac{x^2\,dx}{1+R^2\cos^2(2x)}=\frac{1-q^2}{1+q^2}\left(\frac{\pi^3}{3}+\pi\Re\operatorname{Li}_2\left(iq\right)\right),$$
$$\int_0^\pi\frac{x^2\cos(2x)\,dx}{1+R^2\cos^2(2x)}=\frac{\pi}{R}\frac{1-q^2}{1+q^2}\Im\operatorname{Li}_2(iq),$$

from which (9.35) and (9.36) follow upon using Lemma 5. □

THEOREM 43. *If* r *is a positive integer, then*

$$\frac{1}{F_{2r}\sqrt{5}}\left(\frac{\pi^3}{3}+\frac{\pi}{4}\operatorname{Li}_2\left(-\beta^{2r}\right)\right)=\begin{cases}\int_0^\pi\dfrac{x^2\,dx}{5F_r^2+4\cos^2(2x)}, & \text{if } r \text{ is even};\\ \int_0^\pi\dfrac{x^2\,dx}{L_r^2+4\cos^2(2x)}, & \text{if } r \text{ is odd}.\end{cases} \tag{9.37}$$

PROOF. Set $q=\beta^r$ in (9.35) and use (1.20). □

THEOREM 44. *If r is a positive integer, then*

$$\frac{L_{2r}}{5F_{2r}^3\sqrt{5}}\left(\frac{\pi^3}{3}+\frac{\pi}{4}\operatorname{Li}_2\left(-\beta^{2r}\right)\right)-\frac{\pi}{20F_{2r}^2}\ln\left(1+\beta^{2r}\right)$$
$$=\begin{cases}\int_0^\pi \frac{x^2\,dx}{\left(5F_r^2+4\cos^2(2x)\right)^2}, & \text{if } r \text{ is even;}\\ \int_0^\pi \frac{x^2\,dx}{\left(L_r^2+4\cos^2(2x)\right)^2}, & \text{if } r \text{ is odd.}\end{cases} \tag{9.38}$$

PROOF. Differentiate the Fibonacci-Lucas function form of (9.37). □

THEOREM 45. *If r is a positive even integer, then*

(9.39)
$$\int_0^\pi \frac{x^2\cos(2x)\,dx}{5F_r^2+4\cos^2(2x)}=\frac{\pi}{4L_r}\left(\operatorname{Cl}_2\left(\arctan\left(\frac{2}{F_r\sqrt{5}}\right)\right)+\operatorname{Cl}_2\left(\pi-\arctan\left(\frac{2}{F_r\sqrt{5}}\right)\right)\right)$$
$$-\frac{\pi r}{4L_r}\arctan\left(\frac{2}{F_r\sqrt{5}}\right)\ln\alpha.$$

while, if r is a positive odd integer, then

(9.40)
$$\int_0^\pi \frac{x^2\cos(2x)\,dx}{L_r^2+4\cos^2(2x)}=\frac{\pi}{4F_r\sqrt{5}}\left(\operatorname{Cl}_2\left(\arctan\left(\frac{2}{L_r}\right)\right)+\operatorname{Cl}_2\left(\pi-\arctan\left(\frac{2}{L_r}\right)\right)\right)$$
$$-\frac{\pi r}{4F_r\sqrt{5}}\arctan\left(\frac{2}{L_r}\right)\ln\alpha.$$

PROOF. Set $q=\pm\beta^r$ in (9.36); use (1.20) and Lemma 6. □

10. Results associated with (1.14)

THEOREM 46. *If r is a positive integer, then*

$$\int_0^\pi \frac{x^2\cos x}{L_r-2\cos(2x)}\,dx=\frac{\pi\sqrt{\beta^r}}{1-\beta^r}\left(\operatorname{Li}_2\left(-\sqrt{\beta^r}\right)-\operatorname{Li}_2\left(\sqrt{\beta^r}\right)\right),\quad \text{if } r \text{ is even,} \tag{10.1}$$

(10.2)
$$\int_0^\pi \frac{x^2\cos x}{F_r\sqrt{5}-2\cos(2x)}\,dx=\frac{\pi\sqrt{-\beta^r}}{1+\beta^r}\left(\operatorname{Li}_2\left(-\sqrt{-\beta^r}\right)-\operatorname{Li}_2\left(\sqrt{-\beta^r}\right)\right),\quad \text{if } r \text{ is odd.}$$

In particular,

$$\int_0^\pi \frac{x^2\cos x}{3-2\cos(2x)}\,dx=-\frac{\pi^3}{6}+\frac{3\pi}{2}\ln^2\alpha. \tag{10.3}$$

PROOF. Set $q=\beta^r$ in (1.14) to obtain (10.1) and $q=-\beta^r$ to obtain (10.2). Use (1.20). □

THEOREM 47. *If r is an even positive integer, then*

$$\begin{aligned}\int_0^\pi \frac{x^2\cos x}{\left(L_r-2\cos(2x)\right)^2}\,dx &= -\frac{\pi}{F_r\sqrt{5}}\frac{\sqrt{\beta^r}}{1-\beta^r}\left(\frac{1}{2}+\frac{\beta^r}{1-\beta^r}\right)\left(\operatorname{Li}_2\left(\sqrt{\beta^r}\right)-\operatorname{Li}_2\left(-\sqrt{\beta^r}\right)\right)\\ &\quad-\frac{\pi}{2F_r\sqrt{5}}\frac{\sqrt{\beta^r}}{1-\beta^r}\ln\left(\frac{1+\sqrt{\beta^r}}{1-\sqrt{\beta^r}}\right),\end{aligned}\tag{10.4}$$

while if r is an odd positive integer, then

$$\begin{aligned}\int_0^\pi \frac{x^2\cos x}{\left(F_r\sqrt{5}-2\cos(2x)\right)^2}\,dx &= -\frac{\pi}{L_r}\frac{\sqrt{-\beta^r}}{1+\beta^r}\left(\frac{1}{2}-\frac{\beta^r}{1+\beta^r}\right)\left(\operatorname{Li}_2\left(\sqrt{-\beta^r}\right)-\operatorname{Li}_2\left(-\sqrt{-\beta^r}\right)\right)\\ &\quad-\frac{\pi}{2L_r}\frac{\sqrt{-\beta^r}}{1+\beta^r}\ln\left(\frac{1+\sqrt{-\beta^r}}{1-\sqrt{-\beta^r}}\right).\end{aligned}\tag{10.5}$$

In particular,

$$\int_0^\pi \frac{x^2\cos x}{\left(3-2\cos(2x)\right)^2} = -\frac{\pi}{4}\left(\frac{\pi^2}{3}-3\ln^2\alpha\right)-\frac{3\pi}{2\sqrt{5}}\ln\alpha,\tag{10.6}$$

where we used (9.7).

PROOF. Differentiate the Fibonacci-Lucas function forms of the results in Theorem 46 and take the real part. □

The next results involve the Clausen function.

THEOREM 48. *If r is a positive even integer, then*

$$\begin{aligned}\int_0^\pi \frac{x^2\cos x}{L_r+2\cos(2x)}\,dx &= -\frac{\pi\sqrt{\beta^r}}{1+\beta^r}\left(2\arctan\left(\sqrt{\beta^r}\right)\ln\left(\sqrt{\beta^r}\right)\right)\\ &\quad-\frac{\pi\sqrt{\beta^r}}{1+\beta^r}\left(\operatorname{Cl}_2\left(2\arctan\left(\sqrt{\beta^r}\right)\right)+\operatorname{Cl}_2\left(\pi-2\arctan\left(\sqrt{\beta^r}\right)\right)\right)\end{aligned}\tag{10.7}$$

while if r is a positive odd integer, then

$$\begin{aligned}\int_0^\pi \frac{x^2\cos x}{F_r\sqrt{5}+2\cos(2x)}\,dx &= -\frac{\pi\sqrt{-\beta^r}}{1-\beta^r}\left(2\arctan\left(\sqrt{-\beta^r}\right)\ln\left(\sqrt{-\beta^r}\right)\right)\\ &\quad-\frac{\pi\sqrt{-\beta^r}}{1-\beta^r}\left(\operatorname{Cl}_2\left(2\arctan\left(\sqrt{-\beta^r}\right)\right)+\operatorname{Cl}_2\left(\pi-2\arctan\left(\sqrt{-\beta^r}\right)\right)\right)\end{aligned}\tag{10.8}$$

In particular,

$$\int_0^\pi \frac{x^2\cos x}{3+2\cos(2x)}\,dx = \frac{\pi}{\sqrt{5}}\,\arctan 2\ln\alpha - \frac{\pi}{\sqrt{5}}\,\left(\mathrm{Cl}_2(\arctan 2)+\mathrm{Cl}_2(\pi-\arctan 2)\right), \tag{10.9}$$

where we used $\arctan(-\beta)=\frac{1}{2}\arctan 2$.

Proof. Use of $q=-\beta^r$ in (1.14) produces

$$\int_0^\pi \frac{x^2\cos x}{L_r+2\cos(2x)}\,dx = \frac{\pi i\sqrt{\beta^r}}{1+\beta^r}\left(\mathrm{Li}_2\left(i\sqrt{\beta^r}\right)-\mathrm{Li}_2\left(-i\sqrt{\beta^r}\right)\right),\quad r\text{ even}; \tag{10.10}$$

and hence (10.7) in view of (1.23). Setting $q=\beta^r$ in (1.14) gives

$$\int_0^\pi \frac{x^2\cos x}{F_r\sqrt{5}+2\cos(2x)}\,dx = \frac{\pi\sqrt{\beta^r}}{1-\beta^r}\left(\mathrm{Li}_2\left(\sqrt{\beta^r}\right)-\mathrm{Li}_2\left(-\sqrt{\beta^r}\right)\right),\quad r\text{ odd}, \tag{10.11}$$

which, since $\sqrt{\beta^r}=i\sqrt{-\beta^r}$ for odd r, can also be written as

$$\int_0^\pi \frac{x^2\cos x}{F_r\sqrt{5}+2\cos(2x)}\,dx = \frac{\pi i\sqrt{-\beta^r}}{1-\beta^r}\left(\mathrm{Li}_2\left(i\sqrt{-\beta^r}\right)-\mathrm{Li}_2\left(-i\sqrt{-\beta^r}\right)\right); \tag{10.12}$$

from which (10.8) follows on account of (1.23). □

Theorem 49. *If r is an even positive integer, then*

$$\begin{aligned}\int_0^\pi \frac{x^2\cos x}{\left(L_r+2\cos(2x)\right)^2}\,dx &= -\frac{2\pi}{F_r\sqrt{5}}\frac{\sqrt{\beta^r}}{1+\beta^r}\left(\frac{1}{2}-\frac{\beta^r}{1+\beta^r}\right)\arctan\left(\sqrt{\beta^r}\right)\ln\left(\sqrt{\beta^r}\right)\\ &\quad -\frac{\pi}{F_r\sqrt{5}}\frac{\sqrt{\beta^r}}{1+\beta^r}\left(\frac{1}{2}-\frac{\beta^r}{1+\beta^r}\right)\left(\mathrm{Cl}_2\left(2\arctan\left(\sqrt{\beta^r}\right)\right)-\mathrm{Cl}_2\left(\pi-2\arctan\left(\sqrt{\beta^r}\right)\right)\right)\\ &\quad -\frac{\pi}{2F_r\sqrt{5}}\frac{\sqrt{\beta^r}}{1+\beta^r}\arctan\left(\frac{2\sqrt{\beta^r}}{1-\beta^r}\right),\end{aligned} \tag{10.13}$$

while if r is an odd positive integer, then

$$\begin{aligned}\int_0^\pi \frac{x^2\cos x}{\left(F_r\sqrt{5}+2\cos(2x)\right)^2}\,dx &= -\frac{2\pi}{L_r}\frac{\sqrt{-\beta^r}}{1-\beta^r}\left(\frac{1}{2}+\frac{\beta^r}{1-\beta^r}\right)\arctan\left(\sqrt{-\beta^r}\right)\ln\left(\sqrt{-\beta^r}\right)\\ &\quad -\frac{\pi}{L_r}\frac{\sqrt{-\beta^r}}{1-\beta^r}\left(\frac{1}{2}+\frac{\beta^r}{1-\beta^r}\right)\left(\mathrm{Cl}_2\left(2\arctan\left(\sqrt{-\beta^r}\right)\right)-\mathrm{Cl}_2\left(\pi-2\arctan\left(\sqrt{-\beta^r}\right)\right)\right)\\ &\quad -\frac{\pi}{2L_r}\frac{\sqrt{-\beta^r}}{1-\beta^r}\arctan\left(\frac{2\sqrt{-\beta^r}}{1+\beta^r}\right).\end{aligned} \tag{10.14}$$

Proof. Differentiate the Fibonacci-Lucas function forms of (10.10) and (10.11) and take the real part in each case. □

Lemma 12. *If r is a positive integer, then*

$$\frac{1}{2F_r\sqrt{5}}\int_0^\pi \frac{x^2\cos x}{F_r\sqrt{5}-2\cos(2x)}\,dx+\frac{1}{2F_r\sqrt{5}}\int_0^\pi \frac{x^2\cos x}{F_r\sqrt{5}+2\cos(2x)}\,dx = \int_0^\pi \frac{x^2\cos x}{5F_r^2-4\cos^2(2x)}\,dx \tag{10.15}$$

and

$$\frac{1}{2}\int_0^\pi \frac{x^2\cos x}{F_r\sqrt{5}-2\cos(2x)}\,dx+\frac{1}{2}\int_0^\pi \frac{x^2\cos x}{F_r\sqrt{5}+2\cos(2x)}\,dx = \int_0^\pi \frac{x^2\cos x}{5F_r^2-4\cos^2(2x)}\,dx+\int_0^\pi \frac{x^2\cos(3x)}{5F_r^2-4\cos^2(2x)}\,dx. \tag{10.16}$$

Proof. Identity (10.15) is obvious while (10.16) becomes clear once the elementary trigonometric identity $2\cos x\cos(2x)=\cos(3x)+\cos x$ is employed. □

Theorem 50. *If r is a positive odd integer, then*

(10.17)
$$\begin{aligned}\int_0^\pi \frac{x^2\cos x}{5F_r^2-4\cos^2(2x)}\,dx &= -\frac{\pi}{2F_r\sqrt{5}}\frac{\sqrt{-\beta^r}}{1+\beta^r}\left(\mathrm{Li}_2\left(\sqrt{-\beta^r}\right)-\mathrm{Li}_2\left(-\sqrt{-\beta^r}\right)\right)\\ &\quad-\frac{\pi}{2F_r\sqrt{5}}\frac{\sqrt{-\beta^r}}{1-\beta^r}\left(\mathrm{Cl}_2\left(2\arctan\left(\sqrt{-\beta^r}\right)\right)+\mathrm{Cl}_2\left(\pi-2\arctan\left(\sqrt{-\beta^r}\right)\right)\right)\\ &\quad-\frac{\pi}{F_r\sqrt{5}}\frac{\sqrt{-\beta^r}}{1-\beta^r}\arctan\left(\sqrt{-\beta^r}\right)\ln\left(\sqrt{-\beta^r}\right)\end{aligned}$$

and

(10.18)
$$\begin{aligned}\int_0^\pi \frac{x^2\cos(3x)}{5F_r^2-4\cos^2(2x)}\,dx &= \left(\frac{1}{F_r\sqrt{5}}-1\right)\frac{\pi}{2}\frac{\sqrt{-\beta^r}}{1+\beta^r}\left(\mathrm{Li}_2\left(\sqrt{-\beta^r}\right)-\mathrm{Li}_2\left(-\sqrt{-\beta^r}\right)\right)\\ &\quad+\left(\frac{1}{F_r\sqrt{5}}+1\right)\frac{\pi}{2}\frac{\sqrt{-\beta^r}}{1-\beta^r}\left(\mathrm{Cl}_2\left(2\arctan\left(\sqrt{-\beta^r}\right)\right)+\mathrm{Cl}_2\left(\pi-2\arctan\left(\sqrt{-\beta^r}\right)\right)\right)\\ &\quad+\left(\frac{1}{F_r\sqrt{5}}+1\right)\frac{\pi\sqrt{-\beta^r}}{1-\beta^r}\arctan\left(\sqrt{-\beta^r}\right)\ln\left(\sqrt{-\beta^r}\right).\end{aligned}$$

Proof. Use (10.2) and (10.8) in (10.15) to obtain (10.17). From (10.15) and (10.16) we have

$$\int_0^\pi \frac{x^2\cos(3x)}{5F_r^2-4\cos^2(2x)}\,dx=\left(1-\frac{1}{F_r\sqrt5}\right)\frac12\int_0^\pi \frac{x^2\cos x}{F_r\sqrt5-2\cos(2x)}\,dx$$
$$-\left(1+\frac{1}{F_r\sqrt5}\right)\frac12\int_0^\pi \frac{x^2\cos x}{F_r\sqrt5+2\cos(2x)}\,dx,$$

and hence (10.18) upon using (10.2) and (10.8). □

Lemma 13. *If r is a positive integer, then*

$$\begin{aligned}&\frac{1}{2L_r}\int_0^\pi \frac{x^2\cos x}{L_r-2\cos(2x)}\,dx+\frac{1}{2L_r}\int_0^\pi \frac{x^2\cos x}{L_r+2\cos(2x)}\,dx\\&\quad=\int_0^\pi \frac{x^2\cos x}{L_r^2-4\cos^2(2x)}\,dx\end{aligned}\tag{10.19}$$

and

$$\begin{aligned}&\frac12\int_0^\pi \frac{x^2\cos x}{L_r-2\cos(2x)}\,dx+\frac12\int_0^\pi \frac{x^2\cos x}{L_r+2\cos(2x)}\,dx\\&\quad=\int_0^\pi \frac{x^2\cos x}{L_r^2-4\cos^2 2x}\,dx+\int_0^\pi \frac{x^2\cos(3x)}{L_r^2-4\cos^2(2x)}\,dx.\end{aligned}\tag{10.20}$$

Theorem 51. *If r is a positive even integer, then*

$$\begin{aligned}&\int_0^\pi \frac{x^2\cos x}{L_r^2-4\cos^2(2x)}\,dx\\&\quad=-\frac{\pi}{2L_r}\frac{\sqrt{\beta^r}}{1-\beta^r}\left(\mathrm{Li}_2\left(\sqrt{\beta^r}\right)-\mathrm{Li}_2\left(-\sqrt{\beta^r}\right)\right)\\&\qquad-\frac{\pi}{2L_r}\frac{\sqrt{\beta^r}}{1+\beta^r}\left(\mathrm{Cl}_2\left(2\arctan\left(\sqrt{\beta^r}\right)\right)+\mathrm{Cl}_2\left(\pi-2\arctan\left(\sqrt{\beta^r}\right)\right)\right)\\&\qquad-\frac{\pi}{L_r}\frac{\sqrt{\beta^r}}{1+\beta^r}\arctan\left(\sqrt{\beta^r}\right)\ln\left(\sqrt{\beta^r}\right)\end{aligned}\tag{10.21}$$

and

(10.22)
$$\begin{aligned}&\int_0^\pi \frac{x^2\cos(3x)}{L_r^2-4\cos^2(2x)}\,dx\\&\quad=\left(\frac{1}{L_r}-1\right)\frac{\pi}{2}\frac{\sqrt{\beta^r}}{1-\beta^r}\left(\mathrm{Li}_2\left(\sqrt{\beta^r}\right)-\mathrm{Li}_2\left(-\sqrt{\beta^r}\right)\right)\\&\qquad+\left(\frac{1}{L_r}+1\right)\frac{\pi}{2}\frac{\sqrt{\beta^r}}{1+\beta^r}\left(\mathrm{Cl}_2\left(2\arctan\left(\sqrt{\beta^r}\right)\right)+\mathrm{Cl}_2\left(\pi-2\arctan\left(\sqrt{\beta^r}\right)\right)\right)\\&\qquad+\left(\frac{1}{L_r}+1\right)\frac{\pi\sqrt{\beta^r}}{1+\beta^r}\arctan\left(\sqrt{\beta^r}\right)\ln\left(\sqrt{\beta^r}\right).\end{aligned}$$

In particular,

$$\int_0^\pi \frac{x^2\cos x}{9-4\cos^2(2x)}\,dx = -\frac{\pi}{6}\left(\frac{\pi^2}{6}-\frac{3}{2}\ln^2\alpha\right)+\frac{\pi}{6\sqrt{5}}\arctan 2\ln\alpha - \frac{\pi}{6\sqrt{5}}\left(\mathrm{Cl}_2\left(\arctan 2\right)+\mathrm{Cl}_2\left(\pi-\arctan 2\right)\right).$$

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References

[1] K. Adegoke, The golden ratio, Fibonacci numbers and BBP-type formulas, Fibonacci Quart., 52 (2) (2014), 129–138.
[2] D. Andrica and O. Bagdasar, *Recurrent sequences: Key results, applications, and problems*, Springer, Cham, 2020.
[3] K. Dilcher, Hypergeometric functions and Fibonacci numbers, Fibonacci Quart., 38 (4) (2000), 342–363.
[4] M. L. Glasser and Y. Zhou, An integral representation for the Fibonacci numbers and their generalization, Fibonacci Quart., 53 (4) (2015), 313–318.
[5] I. Gradshteyn and I. Ryzhik, Table of Integrals, Series, and Products, 7th edition, Elsevier Academic Press, Amsterdam, 2007.
[6] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, 2001.
[7] L. Lewin, *Polylogarithms and Associated Functions*, Elsevier/North-Holland, 1981.
[8] S. M. Stewart, Simple integral representations for the Fibonacci and Lucas numbers, Aust. J. Math. Anal. Appl., 19 (2) (2022), Article 2, 5 pages.
[9] S. M. Stewart, A simple integral representation of the Fibonacci numbers, Math. Gaz., 107 (568) (2023), 120–123.
[10] S. B. Tričković and M. S. Stanković, On the closed form of Clausen functions, Integral Transforms Spec. Funct., 34 (6) (2023), 469–477.
[11] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover Press, 2008.

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