SCIENTIA
Series A: *Mathematical Sciences*, Vol. 33 (2023), 35–37
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Uniform convergence is not just for Real Analysis

Victor H. Moll [a] and Supratik Saha [b]

Abstract. The need for uniform convergence is illustrated in the computation of a definite integral for $\zeta(2)$.

1. Introduction

The problem of evaluation of integrals is a remarkable subject. One is introduced to the question in elementary courses, only to be told there that there are some integrals that cannot be computed in terms of elementary functions. The best known example is the primitive of $f(x) = e^{-x^2}$, leading to the introduction of the `error function`

$$\texttt{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt. \tag{1.1}$$

The scaling factor $2/\sqrt{\pi}$ is a normalization based on the value

$$\int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2}, \tag{1.2}$$

a well-known result, useful in Statistic courses.

The lack of a systematic algorithm to evaluate definite integrals was one of the reasons behind the creation of tables of integrals. At the beginning, these were just lists of results which appeared in different scientific problems. With time, these lists grew and they were printed as books growing in size. One of the most popular ones is by I. S. Gradshteyn and I. M. Ryzhik [**2**]. Motivated by a specific example [**1**], the first author of this note, decided to produce proofs of all the entries in this table. The results are still being written in short, self-contained papers, beginning with [**3**].

Recruiting coauthors became an essential part of this project. During the summer of 2022, as part of the `PolyMath Jr. program`, a group of twenty students worked on proving a selected set of entries. Aside from the evaluation themselves, what is interesting is the type of mathematical questions coming from these tasks. This note presents one of them.

2000 *Mathematics Subject Classification.* Primary 26A06, Secondaty 26A09, 11M06.
Key words and phrases. Integrals, zeta(2), uniform convergence, power series.

2. An evaluation

Entry **4.236.2** states that

$$I := \int_0^1 \left[\frac{1}{1-x} + \frac{x \ln x}{(1-x)^2} \right] dx = \frac{\pi^2}{6} - 1. \tag{2.1}$$

The appearance of the value $\zeta(2) = \pi^2/6$ in the answer suggests that the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.2}$$

will appear in the evaluation.

Our first approach began with the series expansions

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{and} \quad \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}. \tag{2.3}$$

The argument proceeds to interchange series and integrals, without worrying about the rigor of the procedure. Naively, we thought, *what can go wrong?* The integral I is then given by

$$\begin{aligned} I &= \int_0^1 \left[\sum_{k=0}^{\infty} x^k + \ln x \sum_{k=1}^{\infty} k x^k \right] dx \\ &= \int_0^1 x^0\, dx + \sum_{k=1}^{\infty} \left[\int_0^1 x^k\, dx + k \int_0^1 x^k \ln x\, dx \right] \\ &= 1 + \sum_{k=1}^{\infty} \left[\frac{1}{k+1} - \frac{k}{(k+1)^2} \right] = 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)^2}, \end{aligned} \tag{2.4}$$

evaluating the second integral using integration by parts. This gives $I = \zeta(2) = \pi^2/6$. *We have lost the* -1 *in the formula.* The first reaction is to suspect that there is a typo in the formula, but numerical and symbolic evaluation of (2.1) produce the stated answer.

In order to provide a correct proof, we proceed with more care. The integral is now evaluated on the interval $[0, a]$, with $0 < a < 1$ fixed, where the power series representations used above converge *uniformly.* The limit as $a \uparrow 1$ will produce the correct value. Indeed, interchanging integrals and summation is valid, and

$$\int_0^a \frac{dx}{1-x} = \sum_{k=0}^{\infty} \int_0^a x^k\, dx = \sum_{k=0}^{\infty} \frac{a^{k+1} - 1}{k+1}, \tag{2.5}$$

and

$$\int_0^a \frac{x}{(1-x)^2} \ln x\, dx = \sum_{k=1}^{\infty} k a^{k+1} \left[\frac{\ln a}{k+1} - \frac{1}{(k+1)^2} \right]. \tag{2.6}$$

Now denote the integral over $[0, a]$ by $I(a)$. The formulas above yield

$$I(a) = (a-1) + \sum_{k=1}^{\infty} \left[\frac{a^{k+1}-1}{k+1} + ka^{k+1} \left(\frac{\ln a}{k+1} - \frac{1}{(k+1)^2} \right) \right] \tag{2.7}$$

and writing k (in the factor ka^{k+1}) as $(k+1)-1$, leads to

$$I(a) = (a-1) + \sum_{k=1}^{\infty} \left[\frac{a^{k+1}-1}{k+1} + a^{k+1}\ln a - \frac{a^{k+1}\ln a}{k+1} - \frac{a^{k+1}\ln a}{k+1} + \frac{a^{k+1}}{(k+1)^2} \right]$$

Letting $a \uparrow 1$ gives

$$\lim_{a \uparrow 1} I(a) = \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} = \zeta(2) - 1, \tag{2.8}$$

as stated in the table.

The evaluation of this entry has been completed and the importance of uniform convergence cannot be ignored.

Acknowledgements. The work presented here is part of the results obtained at the 2022 Polymath Jr. program, partially funded by NSF award DMS-2218374. The authors thank the organizers of this program.

References

[1] G. Boros and V. Moll. An integral hidden in Gradshteyn and Ryzhik. *Jour. Comp. Applied Math.*, 106:361–368, 1999.

[2] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Edited by D. Zwillinger and V. Moll. Academic Press, New York, 8th edition, 2015.

[3] V. Moll. The integrals in Gradshteyn and Ryzhik. Part 1: A family of logarithmic integrals. *Scientia*, 14:1–6, 2007.

Received 13 12 2022 , revised 01 03 2023

[a] Department of Mathematics,Tulane University,
New Orleans, LA 70118, USA
E-mail address: vhm@math.tulane.edu

[b] Institute of Mathematics and Applications, Bhubaneswar, India
E-mail address: supratikmath@gmail.com