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The valuations of power sums

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Abstract. The p-adic valuation of the sequence formed by power sums is described completely for the primes $p = 2$, 3 and 5. A general pattern arising from these explorations is conjectured.

1. Introduction

Given a sequence of positive integers $\{x_n\}$ and a prime p, it is often an interesting question to analyze the sequence formed by looking at the exact power of p dividing x_n. It turns out to be useful to introduce the following notation:

Definition 1.1. Let p be a prime and $x \in \mathbb{N}$. The p-adic valuation of x, denoted by $\nu_p(x)$ is the highest power of p which divides x. That is, x is factored as

$$x = p^{\nu_p(x)} \times b, \quad \text{with } b \text{ not divisible by } p.$$

Information about the sequence $\{\nu_p(x_n)\}$ may include (i) an exact formula; (ii) an asymptotic behavior; (iii) characterization for indices n where $\nu_p(x_n)$ has a specified value. For instance, a classical well-known result of Legendre [**5**] states that

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

A less known result is that Legendre's formula is equivalent to the expression

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1},$$

where $s_p(n)$ is the sum of the base-p digits of n. This version may be used to provide an elementary proof of the fact that the central binomial coefficients $\binom{2n}{n}$ are even numbers and that $\frac{1}{2}\binom{2n}{n}$ is odd precisely when n is a power of 2.

Given a sequence of integers x_n and a prime p, properties of the sequence of valuations $\{\nu_p(x_n)\}$ often present interesting challenges. The problem considered in

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the present work is centered on $\nu_p(S_k(n))$, where

$$S_k(n) = \sum_{j=1}^{k} j^n,$$

is the sum of n^{th}-powers.

The outline of the paper is as follows. Section 2 states some basic properties of the Bernoulli numbers employed in this paper. Section 3 describes the 2-valuation of $S_k(n)$ in complete detail. This is done for the prime $p = 3$ in Section 4. Finally, the analysis of the valuations for the prime $p = 5$ appear in Section 5. Section 6 contains some comments for primes $p \geqslant 7$. Many of the results in the present work have appeared in [**7**].

2. Background material

The sum $S_k(n)$ may be expressed as

$$S_k(n) = \frac{B_{n+1}(k+1) - B_{n+1}}{n+1},$$

where $B_n(x)$ are the Bernoulli polynomials [**6**]. The coefficients B_j are the Bernoulli numbers given by the generating function

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!},$$

and the Bernoulli polynomials are expressed as

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

The above generating function may be used to prove that B_j are rational numbers and that $B_j = 0$ if $j > 1$ is odd. The arithmetic properties of the denominators of B_j, denoted by $\text{Den}(B_j)$, are given by the von Staudt-Clausen Theorem [**4**, Section 7.9, p. 115].

Theorem 2.1. *The denominators of Bernoulli numbers are given by*

$$\text{Den}(B_j) = \prod_{\substack{q \text{ prime} \\ q-1|j}} q.$$

In particular, $\text{Den}(B_j)$ *is always square-free and divisible by 6 when j is even.*

Expanding $B_{n+1}(k+1)$ gives Faulhaber's formula

$$S_k(n) = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n+1}{j} B_j k^{n+1-j}. \tag{2.1}$$

This expression is fundamental to our analysis of the valuations of $S_k(n)$.

Theorem 2.2 stated below gives Kummer's congruences for the numerators of Bernoulli numbers [**3**].

THEOREM 2.2. *Let p be an odd prime, and $\mathbb{Z}_{(p)}$ be the ring of rational numbers whose denominator is prime to p. Then:*

(1) *Suppose n is a positive integer not divisible by $p-1$, Then*

$$\frac{B_n}{n} \in \mathbb{Z}_{(p)}.$$

(2) *Let $a \geqslant 1$ and m, n be even integers such that $a+1 \leqslant m \leqslant n$. Suppose that m and n are not divisible by $p-1$ and $n \equiv m \bmod (p-1)p^{a-1}$. Then we have*

$$\frac{B_n}{n} \equiv \frac{B_m}{m} \bmod p^a$$

In particular, If p be an odd prime greater than 3, then p^l divides $\mathrm{Num}(B_{2.p^l})$.

An application of Theorem 2.2 for *regular* primes is stated below. A prime p is said to be *regular* if p does not divide the numerators of B_{2i} for all $i \in \{1, 2, \ldots \frac{p-3}{2}\}$.

Corollary 2.1. If p is an odd, *regular* prime and n is a positive even integer such that n is not divisible by p and $p-1$, then $\nu_p(B_n) = 0$.

PROOF. Theorem 2.1 implies that p does not divide $\mathrm{Den}(B_n)$. Further, if p is *regular*, then p does not divide $\mathrm{Num}(B_n)$ for all even integers $n = 2, 4, 6, \cdots, p-3$, so $\nu_p(B_n) = 0$. Now we consider the case $n > p-3$ where p and $p-1$ does not divide n, then there exists s even such that $1 < s < p-3 < n$ and $n \equiv s \bmod p-1$. Kummer's congruence now shows

$$B_n \equiv n\frac{B_s}{s} \bmod p \not\equiv 0 \bmod p.$$

Therefore, $\nu_p(B_n) = 0$. □

3. The prime $p = 2$

The main result of this section provides an expression for the 2-adic valuation of $S_k(n)$. The arguments use the formula for $S_k(n)$ in terms of the Bernoulli polynomials given in (2.1). The explicit expression for 2-adic valuation of $S_k(n)$ is presented next. The valuations for some odd primes are given in subsequent sections.

THEOREM 3.1. *For $n > 1$, the 2-adic valuation of $S_k(n)$ is given by*

$$\begin{aligned} \nu_2(S_k(n)) &= \nu_2\left(\left\lfloor \frac{k+1}{2} \right\rfloor\right) \times \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases} \\ &= \nu_2\left(\binom{k+1}{2}\right) \times \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{3.1}$$

PROOF. The argument is divided into cases according to the value of k modulo 4.
$\underline{k \equiv 1,\ 2 \bmod 4}$. This is elementary: $S_k(n) \equiv 1 \bmod 2$, since the sum consists of an odd number of odd summands. The right hand side of (3.1) also vanishes, so the statement holds.

<u>$k \equiv 0 \bmod 4$</u>. Replace k by $4k$, so that now k has no restrictions. The statement to be proven becomes

$$\nu_2(S_{4k}(n)) = \nu_2(2k) \times \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

Consider first the case n is even. Relation (2.1) gives

$$\frac{S_{4k}(n)}{2k} = \frac{1}{n+1}\sum_{j=0}^{n-1}(-1)^j\binom{n+1}{j}(2B_j)2^{2(n-j)}k^{n-j} + 2B_n.$$

Theorem 2.1 shows that $2B_j$ has an odd denominator, therefore every term appearing in the sum is even. The result now follows from the fact that $2B_n$ is odd.

For $n = 1, 3$ the result can be verified directly. For $n \geqslant 5$ odd, (2.1) can be written as

$$\begin{aligned} \frac{S_{4k}(n)}{(2k)^2} &= \frac{2^{2n}}{n+1}k^{n+1} \\ &+ \sum_{j=1}^{n-3}\frac{(-1)^j}{j}\binom{n}{j-1}(2B_j)2^{2(n-j)-1}k^{n-1-j} \\ &+ 2nB_{n-1}. \end{aligned}$$

The first term is even since $\nu_2(n+1) < 2n$, the last term is odd by Theorem 2.1. The next claim completes the proof for the present case.

`Claim`: For $n \geqslant 5$ odd and $1 \leqslant j \leqslant n-3$, the expression

$$a_{n,j} = \frac{1}{j}\binom{n}{j-1}(2B_j)2^{2(n-j)-1} \quad \text{is even.}$$

`Proof of claim`. Write $n = 2m+1$. The case $j = 1$ is clear since $a_{n,1} = 2^{2n-3}$. For $j > 1$, it suffices to take $j = 2t$ even, since the Bernoulli numbers vanish when the index is odd. The claim reduces to proving that

$$b_{m,t} = \frac{1}{t}\binom{2m+1}{2t-1}(2B_{2t})2^{4(m-t)} = \frac{1}{m-t+1}\binom{2m+1}{2t}(2B_{2t})2^{4(m-t)}$$

is even for $1 \leqslant t \leqslant m-1$. This follows directly from the inequalities

$$\nu_2(m-t+1) < 2\log_2(m-t+1) < 4(m-t).$$

The assertion of the claim is now valid.

<u>$k \equiv 3 \bmod 4$</u>. This last case follows from the identity

$$\nu_2(S_{4k-1}(n)) = \nu_2(S_{4k}(n)).$$

To prove this, simply observe that $S_{4k-1}(n) = S_{4k}(n) - (4k)^n$ and use the bounds $\nu_2((4k)^n) = n\,[1+\nu_2(2k)] > 2\nu_2(2k)$ and the expression for the valuation of $S_{4k}(n)$ given above.

The proof of the theorem is complete. $\square$

4. The prime $p = 3$

The analysis of the 3-adic valuation begins with a preliminary lemma.

Lemma 4.1. *Let p be an odd prime. Then*

$$\nu_p(1+(p-1)^n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 2 \\ 1+\nu_p(n) & \text{if } n \equiv 1 \pmod 2 \end{cases} \tag{4.1}$$

Proof. For n even, $1+(p-1)^n \equiv 1+(-1)^n \equiv 2 \pmod p$. If n is odd, we have

$$1+(p-1)^n = \sum_{j=1}^{n} (-1)^{j-1}\binom{n}{j} p^j = pn + p^2 n \sum_{j=2}^{n} \frac{p^{j-2}}{j}(-1)^{j-1}\binom{n-1}{j-1} \tag{4.2}$$

where the last identity comes from the binomial expansion and elementary re-indexing : $\binom{n}{j} = \frac{(n)}{(j)}\binom{n-1}{j-1}$. The result follows from the inequality $\nu_p(j) < j-1$, for $j \geqslant 1$. □

The special case $p = 3$ gives the next result.

Theorem 4.1. *Let $n \in \mathbb{N}$. Then,*

$$\nu_3(S_3(n)) = \begin{cases} 0 & n \text{ is even} \\ \nu_3(n)+1 & n \text{ is odd} \end{cases} \tag{4.3}$$

Proof. The proof comes by applying Lemma 4.1. Note that $S_3(n) = S_2(n) + 3^n$ and $\nu_3(S_2(n))$ is at most $\nu_3(n)+1$ which is less that $\nu_3(3^n) = n$. □

This section presents an explicit formula for the 3-adic valuation of $S_k(n)$. The analysis uses two elementary results given next. The proofs are omitted.

Lemma 4.2. *Assume $x \equiv a \bmod A$ and $x \equiv a \bmod B$, Then $x \equiv a \bmod \operatorname{lcm}(A,B)$.*

Lemma 4.3. *Assume $x \equiv a \bmod A$ and $x \equiv b \bmod B$ with $\gcd(A,B) = 1$. Then*

$$x \equiv aBB^* + bAA^* \bmod AB$$

where $AA^* \equiv 1 \bmod B$ and $BB^* \equiv 1 \bmod A$. Moreover A^*, B^* satisfy $AA^* + BB^* = 1$.

The proof of the main result in this section uses the next auxiliary fact.

Proposition 4.1. *Let n, $k \in \mathbb{N}$. Then $S_k(2n) \equiv kB_{2n} \bmod k$.*

Proof. Use (2.1) to write

$$S_k(2n) = kB_{2n} - \frac{1}{2n+1}\sum_{\ell=3}^{2n+1}(-1)^{\ell}\binom{2n+1}{\ell}B_{2n+1-\ell}k^{\ell}.$$

The result now follows from the congruence:

$$\frac{1}{2n+1}\binom{2n+1}{\ell}B_{2n+1-\ell}k^{\ell} \equiv 0 \bmod k, \quad \text{for } 3 \leqslant \ell \leqslant 2n+1. \tag{4.4}$$

To prove this, let p be a prime dividing k. Then (4.4) follows from

$$\nu_p\left(\frac{1}{2n+1}\binom{2n+1}{\ell}B_{2n+1-\ell}\right)+(\ell-1)\nu_p(k)>0, \tag{4.5}$$

and (4.5) follows from the special case $\nu_p(k)=1$. The elementary identity $\frac{1}{2n+1}\binom{2n+1}{\ell}=\frac{1}{\ell}\binom{2n}{\ell-1}$, shows that it suffices to prove

$$\nu_p\left(\frac{1}{\ell}\binom{2n}{\ell-1}B_{2n+1-\ell}\right)+(\ell-1)>0. \tag{4.6}$$

This is established from the observation that Theorem 2.1 implies that $\nu_p(B_m)\geqslant -1$ and therefore

$$\nu_p\left(\frac{1}{\ell}\binom{2n}{\ell-1}B_{2n+1-\ell}\right)+(\ell-1)\geqslant -\nu_p(\ell)-2+\ell.$$

The inequality required in (4.6) follows from $\nu_p(\ell)\leqslant \log_2\ell$ and an elementary discussion of some special cases for small ℓ. □

The main result is stated next.

THEOREM 4.2. *Let n, $k\in\mathbb{N}$. Then*

$$\nu_3(S_k(n))=\begin{cases}\nu_3(k)+\nu_3(k+1) & \text{if } n=1,\\ \nu_3(k)+\nu_3(k+1)+\nu_3(2k+1)-1 & \text{if } n \text{ is even.}\\ 0 & \text{if } n \text{ is odd and } k\equiv 1 \bmod 3,\\ \nu_3(n)+2\nu_3(k)+2\nu_3(k+1)-1 & \text{if } n>1 \text{ is odd and } k\not\equiv 1 \bmod 3,\end{cases}$$

PROOF. The proof is divided into cases.

$\underline{n=1}$. The result follows from $S_k(1)=\frac{1}{2}k(k+1)$.

$\underline{n \text{ even}}$. Replace n by $2n$, so that now n has no parity restrictions.

`Claim`: $\nu_3(S_k(2n))=\nu_3(S_k(2))$. Then $S_k(2)=\frac{1}{6}k(k+1)(2k+1)$ gives the result.

`Proof of the claim`. Let $x=S_k(2n)$. Replace k by $2k+1$ in the congruence in Proposition 4.1 to obtain $x\equiv(2k+1)B_{2n} \bmod 2k+1$. Using $k+j\equiv-(k-(j-1)) \bmod 2k+1$ gives $2x\equiv(2k+1)B_{2n} \bmod 2k+1$. Now observe that $4k(k+1)\equiv -1 \bmod 2k+1$ to produce $x\equiv -2k(k+1)(2k+1)B_{2n} \bmod 2k+1$. Finally replace k by $k+1$ in the original congruence for x to produce $x\equiv(k+1)B_{2n} \bmod k+1$.

In summary: $x=1^{2n}+2^{2n}+\cdots+k^{2n}$ satisfy the congruences

$$\begin{aligned} x &\equiv k(k+1)(2k+1)B_{2n} \bmod k\\ x &\equiv k(k+1)(2k+1)B_{2n} \bmod k+1\\ x &\equiv -2k(k+1)(2k+1)B_{2n} \bmod 2k+1.\end{aligned}$$

The first two congruences in (4.7) and Lemma 4.2 produce

$$x\equiv k(k+1)(2k+1)B_{2n} \bmod k(k+1).$$

and then the last congruence in (4.7) and Lemma 4.3 (with $A^*=-4$, $B^*=2k+1$) yield

$$x\equiv B_{2n}k(k+1)(2k+1)f(k) \bmod k(k+1)(2k+1)$$

with $f(k) = (2k+1)^2 + 8k(k+1)$. Therefore

$$x \equiv \left(\tfrac{3}{2}B_{2n}\right) \tfrac{2k(k+1)(2k+1)}{3} f(k) \bmod k(k+1)(2k+1).$$

Introduce the notation $t = \frac{2}{3}k(k+1)(2k+1)$ to write the previous congruence in the form

$$\frac{x}{t} = \frac{3}{2}B_{2n}f(k) + \frac{3\alpha}{2}, \quad \text{for some } \alpha \in \mathbb{Z}. \tag{4.7}$$

Theorem 2.1 gives $\nu_3(\frac{3}{2}B_{2n}) = 0$ and since $f(k) \equiv 1 \bmod 3$, it follows from (4.7) that $\frac{x}{t} \not\equiv 0 \bmod 3$. For n even, this yields

$$\nu_3(S_k(n)) = \nu_3(k(k+1)(2k+1)) - 1,$$

as claimed.

<u>n odd and $k \equiv 1 \bmod 3$</u>. Write $n = 2m-1$ and $k = 3t+1$. Then

$$S_k(n) = 1 + \sum_{j=1}^{t} \left((3j-1)^{2m-1} + (3j+1)^{2m-1}\right) + \sum_{j=1}^{t} (3j)^{2m-1}$$

and $(3j-1)^{2m-1} + (3j+1)^{2m-1} \equiv 0 \bmod 3$ shows that $S_k(n) \equiv 1 \bmod 3$. Therefore the valuation of $S_k(n)$ is 0 as claimed.

<u>$n > 1$ odd and $k \equiv 0, 2 \bmod 3$</u>. Replace n by $2n-1$ and observe that (2.1) gives

$$S_k(2n-1) = \frac{1}{2}(2n-1)B_{2n-2}k^2 + \frac{1}{2}\sum_{j=0}^{2n-3}(-1)^j \binom{2n}{j} B_j k^{2n-j}.$$

Thus, one obtains the congruence

$$S_k(2n-1) \equiv \tfrac{1}{2}(2n-1)B_{2n-2}k^2 \equiv \tfrac{1}{2}(2n-1)B_{2n-2}k^2(k+1)^2 \equiv \bmod k.$$

Then $S_{k+1}(2n-1) \equiv S_k(2n-1)$ and the previous argument yields

$$S_k(2n-1) \equiv \tfrac{1}{2}(2n-1)B_{2n-2}k^2(k+1)^2 \bmod k+1.$$

Finally Lemma 4.2 gives $S_k(2n-1) \equiv \frac{1}{2}(2n-1)B_{2n-2}k^2(k+1)^2 \bmod k(k+1)$, and writing this as

$$S_k(2n-1) \equiv \left(\tfrac{3}{2}B_{2n-2}\right)(2n-1) \times \tfrac{1}{3}k^2(k+1)^2 \bmod k(k+1),$$

the result follows as in the previous case (when n is even). The proof is now complete. □

Remark 4.1. The formula presented in Theorem 4.2 can be re-stated in terms of the base cases $S_k(n)$ for $n = 1, 2, 3$. This gives a reduction formula for the p-adic valuation which is extended to general prime case in Section 6.

$$\nu_3(S_k(n)) = \begin{cases} \nu_3(S_k(1)) & n = 1 \\ \nu_3(S_k(2)) & n \text{ is even} \\ 0 & n \text{ is odd, } k \equiv 1 \pmod 3 \\ \nu_3(S_3(n)) + \nu_3(S_k(3)) - 2 & n > 1 \text{ is odd, } k \not\equiv 1 \pmod 3. \end{cases} \tag{4.8}$$

Notice from this formula that when n is even, $\nu_3(S_k(n))$ is actually independent of the value of n, implying that when $k \equiv 1 \pmod 3$, $\nu_3(S_k(n)$ oscillates between 0 and $\nu_3(S_k(2))$ depending on whether $n > 1$ is even or odd.

Corollary 4.1. For $n,\ k \in \mathbb{N}$,

$$\nu_3(S_{3k}(n)) = \nu_3(S_{3k-1}(n)).$$

PROOF. The proof comes directly from the formula in Theorem 4.2. $\square$

5. The prime $p = 5$

This section presents an analysis of the valuations $\nu_5(S_k(n))$. Complete results are given for $2 \leqslant k \leqslant 5$ where there are explicit `analytic` expressions for these valuations. In the case $k = 6$ the valuation is given in terms of a tree structure. Such a phenomena has appeared in the valuations of Stirling numbers of the second kind [**2**] and also in the valuations of polynomial sequences [**1**].

5.1. The case $k = 2$. The 5-adic valuation of $S_2(n)$ is divided according to the residue $n \bmod 4$. The valuation itself is given in terms of the digits of the base 5-expansion of n; that is, in the expansion

$$n = a_0 + a_1 \cdot 5 + a_2 \cdot 5^2 + a_3 \cdot 5^3 + \cdots .$$

The next result comes directly from 4.1.

LEMMA 5.1. *For $n \in \mathbb{N}$,*

$$\nu_5(1 + 2^n) = \begin{cases} 0 & \text{if } n \not\equiv 2 \bmod 4 \\ 1 + \nu_5(n) & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

The next theorem states the 5-adic valuation of $S_2(n)$ in terms of the base 5 expansion of n. This is done by introducing the following notation: for $n \equiv j \bmod 5$. Define $x(n, j)$ to be the length of the initial segment of j's in the base 5-expansion of n (this is a segment composed of $x(n, j)$ copies of the integer j).

THEOREM 5.1. *Assume the base 5 expansion of n starts with a sequence of $r + 1$ indices equal to 2; that is, $a_0 = a_1 = \cdots = a_r = 2$ and $a_{r+1} \neq 2$. We say $x(n, 2) = r + 1$. Then*

$$\nu_5(S_2(4n + j)) = \begin{cases} 0 & \text{if } j \not\equiv 2 \bmod 4 \\ x(n, 2) + 1 & \text{if } j \equiv 2 \bmod 4. \end{cases}$$

PROOF. Start with $S_2(4n + j) = 1 + 2^{4n+j} \equiv 1 + 2^j \bmod 5$. Then $S_2(4n + j) \not\equiv 0 \bmod 5$ when $j \not\equiv 2 \bmod 4$. This gives the valuation for $j \not\equiv 2 \bmod 4$.

Now assume $j \equiv 2 \bmod 4$, so that $a_0 = 2$, and write $n = \displaystyle\sum_{k=0}^{r} 2 \cdot 5^k + \sum_{k \geqslant r+1} a_k 5^k$, with $a_{r+1} \neq 2$. Then $4n + 2 = (2 + 4a_{r+1})\, 5^{r+1} + \displaystyle\sum_{k \geqslant r+2} 4a_k 5^k$. Since $a_{r+1} \neq 2$, it follows that $2 + 4a_{r+1}$ is not divisible by 5. Therefore $\nu_5(4n + 2) = r + 1$. The proof now follows from Lemma 5.1. $\square$

Remark 5.1. Note that the work in this proof can be generalized as follows

$$\nu_p\left((p-1)\,n+j\right) = x(n,j) \text{ when } j|(p-1) \tag{5.1}$$

5.2. The case $k=3$. The 5-adic valuations of $S_3(n)$ are easily accessible.

Theorem 5.2. *For any $n \in \mathbb{N}$,*

$$\nu_5(S_3(n)) = 0.$$

Proof. Observe that $S_3(n) = 1 + 2^n + 3^n \equiv 1 + 2^n(1+(-1)^n) \bmod 5$. Therefore $S_3(n) \equiv 1 \bmod 5$ for n odd. For $n = 2t$, then $S_3(n) = 1 + 2 \cdot 4^t$. This expression has values 3 or 4 mod 5. Therefore $S_3(n) \not\equiv 0 \bmod 5$ and the result follows from here. □

5.3. The case $k=4$. The 5-adic valuations of $S_4(n)$ is discussed next according to the residue of n modulo 4. The case $n \equiv 0 \bmod 4$ is easy. Each of the other three cases admit similar proofs, the details are given for $n \equiv 2 \bmod 4$ only.

$n \equiv 0 \bmod 4$. In this case, replacing n by $4m$,

$$S_4(4m) = 1^{4m} + 2^{4m} + 3^{4m} + 4^{4m} = 1 + 16^m + 81^m + 256^m \equiv 4 \bmod 5,$$

and it follows that $\nu_5(S_4(4m)) = 0$.

$n \equiv 1 \bmod 4$. Now write $n = 4m+1$. Then

$$\nu_5(S_4(4m+1)) = x(m,1) + 1.$$

Proof. Write the sum $S_4(4n+1) = 1 + 4^n + 2^n + (5-3)^n$. Since n is odd, by Lemme 4.1 we have

$$1 + (4)^n = \sum_{j=1}^{n} (-2)^{n-j} \binom{n}{j} 5^j = 5n + 5^2 n \sum_{j=2}^{n} \frac{5^{j-2}}{j} (-1)^{n-j} \binom{n-1}{j-1}. \tag{5.2}$$

Similarly, we get

$$2 + (5-2)^n = \sum_{j=1}^{n} (-2)^{n-j} \binom{n}{j} 5^j = 5 \cdot 2n + 5^2 n \sum_{j=2}^{n} \frac{5^{j-2}}{j} (-2)^{n-j} \binom{n-1}{j-1}. \tag{5.3}$$

So we have,

$$\begin{aligned} S_4(n) &= 5n + 10n + 5^2 n \sum_{j=2}^{n} \frac{5^{j-2}}{j} \binom{n-1}{j-1} ((-1)^{n-j} + (-2)^{n-j}) \\ &= 5n \left(3 + 5 \sum_{j=2}^{n} \frac{5^{j-2}}{j} \binom{n-1}{j-1} ((-1)^{n-j} + (-2)^{n-j}) \right) \end{aligned}$$

Since $n = 4m+1, \nu_5(S_4(n) = 1 + \nu_5(n) = x(m,1) + 1$. □

THEOREM 5.3. *For $n \in \mathbb{N}$ and assume $n \equiv j \mod 5$. Recall that $x(n,j)$ is the length of the initial segment of j's in the base 5-expansion of n. Then*

$$\nu_5(S_4(4m+j)) = \begin{cases} 0 & \text{if } j=0 \\ 1+x(m,j) & \text{if } j=1 \text{ or } j=2, \\ 2+x(m,3) & \text{if } j=3 \text{ and } m \not\equiv 2 \pmod 5 \\ 2+x(m,2) & \text{if } j=3 \text{ and } m \equiv 2 \pmod 5 \end{cases}$$

The next examples illustrate the result in Theorem 5.3.

$m=3$ and $j=0$. Then

$$S_4(4\cdot 3) = 17312754 = 2\cdot 3\cdot 97\cdot 151\cdot 197$$

shows that $\nu_5(S_4(4m+j)) = 0$ if $j=0$.

$m=6$ and $j=1$. Then

$$S_4(4\cdot 6+1) = 2^2\cdot 5^3\cdot 97\cdot 103\cdot 225552443,$$

so that $\nu_5(S_4(4\cdot 6+1)) = 3$. The expansion $6 = 1\cdot 5^0 + 1\cdot 5^1$ shows that $x(6,1)=1$ and this is consistent with the statement in the theorem.

$m = 37 \equiv 2 \pmod 5$ and $j=3$. Then

$$S_4(4\cdot 37+3) = 2^2\cdot 5^4\cdot 17\cdot 101\cdot \text{ a prime factor of 82 digits.}$$

The expansion $37 = 2\cdot 5^0 + 2\cdot 5^1 + 1\cdot 5^2$ shows that $x(n,2)=2$ and it confirms that $\nu_5(S_4(4\cdot 37+3) = 4$ as stated in the theorem.

5.4. The case $k=5$. The 5-adic valuations of $S_5(n)$ reduces to $S_4(n)$ in an elementary manner.

THEOREM 5.4. *For $n \in \mathbb{N}$,*

$$\nu_5(S_5(n)) = \nu_5(S_4(n)).$$

PROOF. This follows directly from $S_5(n) = S_4(n) + 5^n$. □

5.5. The case $k=6$. This is the first example in which the valuation $\nu_5(S_6(n))$ is given by a *tree structure*. Start with the expression

$$S_6(n) = 1^n + 2^n + 3^n + 4^n + 5^n + 6^n$$

and recall that the valuation of an integer x is determined by the highest power of the prime p with the condition $x \equiv 0 \mod p^r$ and $x \not\equiv 0 \mod p^{r+1}$. Therefore it is convenient to analyze $S_6(n)$ modulo powers of 5.

Step 1: $S_6(n)$ modulo 5. A direct calculation gives

$$\begin{aligned} S_6(n) &\equiv 1^n + 2^n + (-2)^n + (-1)^n + 0 + 1^n \mod 5 \\ &\equiv 2 + (-1)^n + [1+(-1)^n]\, 2^n \mod 5. \end{aligned}$$

Therefore, if n is odd, then $S_6(n) \equiv 1 \mod 5$ and for n even, we have two cases: if $n \equiv 2 \mod 4$, then $S_6(n) \equiv 3 + 2^{4t+3} \mod 5 \equiv 3 + 16^t\cdot 8 \equiv 1 \not\equiv 0 \mod 5$, and and if $n \equiv 0 \mod 4$, then $S_6(n) \equiv 0 \mod 5$. This proves

Lemma 5.2. *Let* $n \in \mathbb{N}$. *Then*

$$\nu_5(S_6(n)) = \begin{cases} 0 & \text{if } n \not\equiv 0 \bmod 4 \\ \geqslant 1 & \text{if } n \equiv 0 \bmod 4. \end{cases}$$

Step 2: Take $n \equiv 0 \bmod 4$, write $n = 4n_0$ and consider n_0 modulo 5 by writing $n_0 = 5n_1 + j_1$, with $0 \leqslant j_1 \leqslant 4$. Then $S_6(n) \equiv 0 \bmod 5$ and

$$S_6(n) = S_6(4 \cdot 5n_1 + 4j_1) = \sum_{k=1}^{6} k^{4 \cdot 5n_1} k^{4j_1}.$$

To evaluate this sum modulo 5^2, observe the term with $k = 5$ vanishes and the remaining ones satisfy $k^{4 \cdot 5n_1 + 4j_1} \equiv (k^{4 \cdot})^{n_1} \times k^{4j_1} \equiv k^{4j_1} \bmod 5^2$, in view of Euler's formula $x^{\varphi(5^2)=4\cdot 5} \equiv 1 \bmod 5^2$. It follows that $S_6(n) \equiv S_6(4j_1) \bmod 5^2$. Computing these five values, gives

Lemma 5.3. *Let* $n \in \mathbb{N}$ *be congruent to* 0 *modulo* 4. *Write* $n = 4(5n_1 + j_1)$, *then* $\nu_5(S_6(n) \geqslant 1$ *and*

$$\nu_5(S_6(n)) = \begin{cases} 1 & \text{if } n_1 \in \mathbb{N} \text{ and } j_1 \neq 1 \\ \geqslant 2 & \text{if } n_1 \in \mathbb{N} \text{ and } j_1 = 1. \end{cases}$$

Step 3: Indices of the form $n = 4(5n_1 + 1)$, for which $\nu_6(S_6(n)) \geqslant 2$ are now split by considering n_1 in classes modulo 5; that is, write $n_1 = 5n_2 + j_2$ and compute $S_6(n)$ modulo 5^3. As in the previous step

$$S_6(n) = S_6(4 \cdot 5^2 n_1 + 4 \cdot 5j_2 + 4) = \sum_{k=1}^{6} k^{4 \cdot 5^2} \cdot k^{4 \cdot 5j_2} \cdot k^4$$

and since $k^{4\cdot 5^2} \equiv 1 \bmod 5^2$ for $k \neq 5$ and $\equiv 0 \bmod 5^3$ for $k = 5$, it follows that $S_6(n) \equiv S_6(4 \cdot 5j_2 + 4) \bmod 5^3$. A direct evaluation of the five sums $S_6(4 \cdot 5j_2 + 4)$ modulo 5^3, for $0 \leqslant j_2 \leqslant 4$, shows that only $j_2 = 1$ gives zero modulo 5^3. This proves

Lemma 5.4. *Let* $n \equiv 0 \bmod 4$ *be of the form* $n = 4(5^2 n_2 + 5j_2 + 1)$, *with* $n_2 \in \mathbb{N}$ *and* $0 \leqslant j_2 \leqslant 4$. *Then* $\nu_5(S_6(n)) \geqslant 2$ *and*

$$\nu_5(S_6(n)) = \begin{cases} 2 & \text{if } n_2 \in \mathbb{N} \text{ and } j_2 \neq 1 \\ \geqslant 3 & \text{if } n_2 \in \mathbb{N} \text{ and } j_2 = 1. \end{cases}$$

It is now conjectured that this procedure may be continued indefinitely. This determines a tree structure for the 5-adic valuation of $S_6(n)$.

Conjecture 5.1. Assume that the procedure described above has been completed r times. This determines integers $j_1, j_2, \ldots, j_{r-1}$, all in the set $\{0, 1, 2, 3, 4\}$, such that

$$n = 4(5^{r+1} n_{r+1} + 5^r j_r + r^{r-1} j_{r-1} + \cdots + 5j_2 + j_1)$$

so that the condition $\nu_2(S_6(n)) \geqslant r$ is satisfied. It is conjectured that **exactly one** of the values $j_r \in \{0, 1, \cdots, 4\}$, called the `non-terminal index` satisfies $S_6(n)(n) \equiv$

0 mod 5^r, while the other four (called `terminal indices`) produce $S_6(n) \not\equiv 0 \bmod 5^r$. Therefore

$$\nu_5(S_6(n)) = \begin{cases} r & \text{if } j_r \text{ is terminal} \\ \geqslant r+1 & \text{if } j_r \text{ is not terminal.} \end{cases}$$

6. Comments on higher primes

This section discusses, for *regular prime* $p \geqslant 7$, some special cases of $\nu_p(S_k(n))$. The results presented here are consequences of Faulhaber's formula (2.1) and properties of the Bernoulli numbers given in Section 2.

The valuation of power sums is now connected with that of the Bernoulli numbers.

THEOREM 6.1. *If $n \in \mathbb{N}$ and p an odd, regular prime, then*

$$\nu_p(S_p(n)) = \begin{cases} \nu_p(B_n) + 1 & n \text{ is even} \\ \nu_p(n) + \nu_p(B_{n-1}) + 2 & n \text{ is odd.} \end{cases}$$

PROOF. Faulhaber's formula (2.1) gives

$$\begin{aligned} S_p(n) &= \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i \binom{n+1}{i} B_i p^{n+1-i} \\ &= \frac{1}{n+1} \left(B_0 p^{n+1} - B_1 (n+1) p^n \ldots \frac{(-1)^{n-1} B_{n-1} (n+1)(n) p^2}{2} + (-1)^n B_n (n+1) p \right). \end{aligned}$$

Each term in this sum has distinct p-adic valuations and thus

$$\nu_p(S_p(n)) = \min_{0 \leqslant i \leqslant n} \left\{ \nu_p \left(\binom{n+1}{i} B_i p^{n+1-i} \right) \right\}$$

is determined by the last term of the sum. For n even, the last term is pB_n and for n is odd, $B_n = 0$ so the last term is now $np^2 B_{n-1}$. This confirms the result. □

REMARK 6.1. Applying 5.3 and 6.1 to $p = 5$ we can relate the 5-adic expansion of m to the 5-adic valuation of B_{4m+2}, in particular

$$\nu_5(B_{4m+2}) = \nu_5(4m+2) = x(m, 2)$$

.

Corollary 6.1. Let $n \in \mathbb{N}$ and $p \geqslant 3$ an regular odd prime. Then

$$\nu_p(S_p(n)) = 0 \text{ if and only if } n \equiv 0 \bmod p-1.$$

PROOF. The proof is a direct consequence of Theorems 6.1 and 2.1. □

Corollary 6.2. Let $n \in \mathbb{N}$ and p be regular prime such that $2n \not\equiv 0 \bmod p-1$. Then,

$$\nu_p(S_p(2n)) = \nu_p(\text{Num}(B_{2n}) + 1, \text{ and}$$
$$\nu_p(S_p(2n+1)) = \nu_p(\text{Num}(B_{2n})) + \nu_p(2n+1) + 2$$

Proof. If $2n$ is not divisible by $p-1$, Theorem 2.1 shows that p does not divide the denominator of B_{2n}. It follows that the p-adic valuation of the Bernoulli number in Theorem 6.1 is the same as the p-adic valuation of the numerator of the Bernoulli number. The result follows from here. □

Corollary 6.2 relates the p-divisibility of the power sums to the p-divisibility of the numerators of Bernoulli numbers. By Kummer's Congruence, we get $\nu_p(S_p(2p^l)) = \nu_p(\text{Num}(B_{2p^l}) + 1 \geqslant l+1$. On the other hand, it also implies than if p^2 divides $S_p(n)$ then p divides $Num(B_{2n})$.

Remark 6.2. Corollary 6.2 implies the recurrence relation

$$\nu_p(S_p(2n+1)) = \nu_p(S_p(2n)) + \nu_p(2n+1) + 1.$$

Corollary 6.1 classifies indices n for which $\nu_p(S_p(n)) = 0$. The next result partially classifies indices for which $\nu_p(S_p(n)) = 1$.

Corollary 6.3. Let p be an odd prime p. Then

(1) If p is *regular* prime, then $\nu_p(S_p(2n)) = 1$ if and only if $2n \not\equiv 0 \bmod p$ and $2n \not\equiv 0 \bmod p-1$
(2) $\nu_p(S_p(2n+1)) = 1$ if and only if $2n+1 \not\equiv 0 \bmod p$ and $2n \equiv 0 \bmod p-1$.

Proof. The proof is a direct consequence of Theorem 6.1 and Corollary 2.1. First assume n is even and write $2n$ instead of n. Then $\nu_p(S_p(2n)) = 1$ if and only if $\nu_p(B_{2n}) = 0$. Corollary 2.1 holds precisely when p is *regular* and $2n$ is not divisible by p nor $p-1$.
In the case n odd, replace n by $2n+1$. Then $\nu_p(S_p(2n+1)) = \nu_p(B_{2n}) + \nu_p(2n+1) + 2 = 1$ if and only if $\nu_p(B_{2n}) = -1$ and $\nu_p(2n+1) = 0$, i.e $p-1$ divides $\text{Den}(B_{2n})$ and p does not divide $2n+1$. The result now follows from Theorem 2.1. □

The next result gives the special case when k is a power of p. The result separates the variables n and k. This also holds for $p = 2$ and 3.

Theorem 6.2. *Let $n, k \in \mathbb{N}$ and p an odd, regular prime such that $k = p^s$; that is, $\nu_p(k) = s$. Then*

$$\nu_p(S_k(n)) = \nu_p(S_{p^s}(n)) = \begin{cases} \nu_p(k) & n = 1 \\ \nu_p(S_p(n)) + 2(\nu_p(k) - 1) & n \textit{ is odd} \\ \nu_p(S_p(n)) + (\nu_p(k) - 1) & n \textit{ is even} \end{cases}$$

PROOF. The proof is split into cases depending on n. The case $n = 1$ is clear from expanding $S_k(1)$. The case n odd is considered first. Faulhaber's formula (2.1), Theorem 6.1 and $B_n = 0$, it follows that

$$\begin{aligned}\nu_p(S_{p^s}(n)) &= \nu_p(B_{n-1}np^{2s}) \\ &= \nu_p(B_{n-1}np) + \nu_p(2s-2) \\ &= \nu_p(S_p(n)) + 2(\nu_p(k)-1).\end{aligned}$$

The case n even is similar. □

THEOREM 6.3. *For any n, $s \in \mathbb{N}$ and l co-prime to regular prime p,*

$$\nu_p(S_{p^s l}(n)) = \nu_p(S_{p^s}(n))$$

PROOF. Apply Faulhaber's formula (2.1) to $S_{p^s l}(n)$:

$$\begin{aligned}S_{p^s l}(n) &= \frac{1}{n+1}\sum_{i=0}^{n}(-1)^i\binom{n+1}{i}B_i(p^s)^{n+1-i}l^{n+1-i} \\ &= \frac{1}{n+1}\left(B_0p^{s(n+1)}l^{n+1} + \cdots + \frac{(-1)^{n-1}B_{n-1}(n+1)(n)p^{2s}l^2}{2} + (-1)^nB_n(n+1)p^sl\right).\end{aligned}$$

The p-adic valuation of this sum is given by the p-adic valuation of the last or the second to last term, depending on the parity of n. The result now follows from $\nu_p(l) = 0$. □

This section concludes by proposing a conjecture generalizing of Corollary 4.1. This conjecture is shown to be equivalent to a bound on $\nu_p(S_{pm}(n))$.

CONJECTURE 6.1. For any regular prime p and $n \in \mathbb{N}$, then

$$\nu_p(S_{pm-1}(n)) = \nu_p(S_{pm}(n)). \tag{6.1}$$

Since $S_{pm-1}(n) = S_{pm}(n) - (pm^n)$, it follows that

$$\nu_p(S_{pm-1}(n)) \geqslant \min\{\nu_p(S_{pm}(n)), \nu_p((pm)^n)\}.$$

Thus equation (6.1) is equivalent to

$$\nu_p(S_{pm}(n)) \leqslant \nu_p((pm)^n).$$

This has been verified on `Mathematica` up to $p = 229$ (the 50^{th} prime) and $m \leqslant 100$, $n \leqslant 100$. It is possible to show that Conjecture 6.1 holds for $n = 1$. Theorems 3.1 and 4.2 imply that it holds for $p = 2$ and $p = 3$. Corollary 6.1 may then be used to prove this conjecture when n is even and $p-1|n$, since under these conditions $\nu_p(S_{(p-1)l}(n) = 0$. On the other hand, Corollary 2.1 and Kummer's congruence show that equation (6.1) holds if p is a regular prime and p and $p-1$ do not divide n. The general case has eluded us.

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