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Review of some iterative root–finding methods from a dynamical point of view

Sergio Amat*, Sonia Busquier, and Sergio Plaza‡

Abstract. From a dynamical point of view applied to complex polynomials, we study a number of root–finding iterative methods. We consider Newton's method, Newton's method for multiple roots, Jarratt's method, the super–Halley method, the convex as well as the double convex acceleration of Whittaker's method, the methods of Chebyshev, Stirling, and Steffensen, among others. Since all of the iterative root–finding methods we study satisfy the Scaling Theorem, except for Stirling's method and that of Steffensen, we obtain their conjugacy classes.

1. Introduction

Before formulating the problems we choose to investigate, we recall some basic notions of complex dynamics. Let $R : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ be a rational map on the Riemann sphere, that is, $R(z) = \frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are polynomials without common factors. The degree of $R(z)$ is defined as $\deg(R) = \max\{\deg(p), \deg(q)\}$. In what follows, we will consider only rational maps of degree greater than or equal to two.

Let R be a rational map. For $z \in \overline{\mathbb{C}}$ we define its orbit as the set

$$\mathrm{orb}(z) = \{z, R(z), \ldots, R^{\circ k}(z), \ldots\},$$

where $R^{\circ k}$ stand for the k–fold iterate of R. A point z_0 is a fixed point of R if $R(z_0) = z_0$. A periodic point of period n is a point z_0 such that $R^{\circ n}(z_0) = z_0$ and $R^{\circ j}(z_0) \neq z_0$ for $0 < j < n$. Observe that if $z_0 \in \overline{\mathbb{C}}$ is a periodic point of period $n \geqslant 1$, then it is a fixed point of $R^{\circ n}$. A fixed point z_0 of R is *attracting, repelling*, or *indifferent* if $|R'(z_0)|$ is less than, greater than, or equal to 1, respectively. A *superattracting fixed point* of R is a fixed point which is also a critical point of R. A periodic point of period n is *attracting, superattracting, repelling*, or *indifferent* if it is, as a fixed point of $R^{\circ n}$, attracting, superattracting, repelling, or indifferent, respectively. The *Julia* set of a rational map R, denoted $\mathcal{J}(R)$, is the closure of the

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set consisting of its repelling periodic points. Its complement is the *Fatou set*, denoted $\mathcal{F}(R)$.

Let ζ be an attracting fixed point of R. Its *basin of attraction* is the set $B(\zeta) = \{z \in \overline{\mathbb{C}} : R^{\circ n}(z) \longrightarrow \zeta \text{ as } n \longrightarrow \infty\}$. The immediate basin of attraction of an attracting fixed point ζ of $R(z)$, denoted $B^*(\zeta)$, is the connected component of $B(\zeta)$ containing ζ. Finally, if z_0 is an attracting periodic point of period n of R, then the basin of attraction of the orbit $\text{orb}(z_0)$ is the set $B(\text{orb}(z_0)) = \cup_{j=0}^{n-1} R^{\circ j}(B(z_0))$, where $B(z_0)$ is the attraction basin of z_0 as a fixed point of $R^{\circ n}$, and its immediate basin of attraction is the set $B^*(\text{orb}(z_0)) = \cup_{j=0}^{n-1} R^{\circ j}(B^*(z_0))$. If R has an attracting periodic point z_0, then the basin of attraction is contained in the Fatou set and $\mathcal{J}(R) = \partial B(z_0)$, which is the topological boundary of $B(z_0)$. Therefore, the chaotic dynamics of R is contained in its Julia set.

For an extensive and comprehensive review of the theory of iteration of rational maps, see [**13**] and [**14**]. For a general treatment of the theory of iteration of rational maps, see [**38**], [**39**], [**32**], [**10**], [**53**]. For a more advanced approach to the theory of iteration of rational maps, see [**40**] and [**37**]. As far as the history of complex dynamics is concerned, the book by D. Alexander [**1**] (and references therein) is a quite valuable source.

Newton's iterative method associated to an analytic function $f(z)$ is $N_f(z) = z - \frac{f(z)}{f'(z)}$. Now if $p(z)$ is a complex polynomial, then the function $N_p(z)$ defines a rational map on the Riemann sphere $\overline{\mathbb{C}}$, and hence defines a discrete dynamical system $z_{n+1} = N_p(z_n)$. If α is a simple root of p (that is, $p(\alpha) = 0$ and $p'(\alpha) \neq 0$), then α is a superattracting fixed point of N_p and, generically, $N_p''(\alpha) \neq 0$. Consequently, if the initial guess z_0 is chosen near α, then the sequence of iterates $(z_n)_{n\in\mathbb{N}}$ converges quadratically to α, or in other words $|z_{n+1} - \alpha| \leqslant c|z_n - \alpha|^2$ for some constant $c > 0$. This fact makes Newton's method one of the most widely used methods for approximating the roots of polynomials. In [**30**], F. v. Haeseler and H-O. Peitgen discuss the dynamics of Newton's algorithm and give a description of the basins of attraction of the roots. In [**57**] and [**58**], the dynamics of the families of iterative methods of Schröder and of König is addressed, and the parameter spaces of both families of methods associated to the one–parameter family of cubic polynomials $p_A(z) = z^3 + (A-1)z - A$ are described in the case the order of convergence is either three or four. An analogous study for Newton's iterative method associated to the one–parameter family $p_A(z)$ above was begun by J. H. Curry, L. Garnett and D. Sullivan. In this work, parameter regions in which extraneous attractive periodic cycles exist are described. (See [**20**].) A similar feature is also observed for Schröder's family of iterative methods associated to $p_A(z)$, as well as for König's family associated to $p_A(z)$. (See [**57**], [**58**].) For a more recent study on the subject for higher orders of convergence, see [**8**] and [**7**], as well as [**22**]. Another well known iterative root–finding method is Halley's iterative method; for a study of its dynamics for real functions, see [**12**] and the references therein. In [**51**], the geometry of Halley's method is studied. In [**16**], recent advances in a description of the conjugacy classes and the dynamics of König's family of iterative root–finding algorithms applied to complex polynomials, which reduce to Newton's and Halley's methods for order 2 and 3, respectively, are given. Advances on the study of the dynamics for the super–Newton method, as well as for Cauchy's and Halley's methods are carried out in [**35**].

2. Basic definitions and results

In what follows, we will assume that $f : U \longrightarrow \mathbb{C}$ is an analytic function, where $U \subset \mathbb{C}$ is an open set. We focus on the case $U = \mathbb{C}$ and $f(z)$ a polynomial function.

DEFINITION 2.1. *We say that a map* $f \longrightarrow T_f$ *carrying a complex–valued function* $f(z)$ *to a function* $T_f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ *is an iterative root–finding algorithm if* $T_f(z)$ *has a fixed point at every root of* $f(z)$*, and given an initial guess* z_0*, the sequence of iterates* $(z_k)_{k\geqslant 0}$*, where* $z_{k+1} = T_f(z_k)$*, converges to a root* $r \in \overline{\mathbb{C}}$ *of* $f(z)$ *whenever* z_0 *is sufficiently close to* r*.*

Cayley's Problem

In his study on the convergence of Newton's iterative map, A. Cayley poses the following question: Let $p(z)$ be a polynomial. *What is the set consisting of the initial guesses* $z_0 \in \mathbb{C}$ *for which the sequence of iterates* $z_n = N_p(z_{n-1})$*, with* $n \geqslant 1$*, converges to a root* α *of* $p(z)$*. In other words, what is the basin of attraction of* α*?* (See [**17**], as well as [**18**].)

We can ask the same question for an arbitrary iterative root–finding method $T_p : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$. Now it is clear that $\mathcal{F}(T_p) \supset \cup_{j=1}^{k} B(r_j)$, where $r_1, \ldots, r_k$ are the roots of $p(z)$. It is natural to ask the following questions. Let $p(z)$ be a polynomial. *What is the set consisting of the initial guesses* $z_0 \in \mathbb{C}$ *for which the sequence of iterates* $z_{n+1} = T_p(z_n)$*, with* $n \geqslant 1$*, converges to a root* α *of* $p(z)$*. In other words, what is the basin of attraction of* α*? What is the set consisting of the points* z_0 *such that the sequence of iterates* $z_{n+1} = T_p(z_n)$ *does not converge to any root of* $p(z)$*?* Now since $\mathcal{J}(T_p)$ is an invariant set, we have that if $z_0 \in \mathcal{J}(T_p)$, then its orbits will be contained in $\mathcal{J}(T_p)$. Since $\mathcal{J}(T_p)$ is the closure of repelling periodic points of T_p, and since in computer experiments we only use arithmetic of finite precision, we have that if $z_0 \in \mathcal{J}(T_p)$, then z_n will eventually be thrown off $\mathcal{J}(T_p)$ due to roundoff error (even if small). Now if there is an (super)attracting periodic orbit of period greater than or equal to two, denoted $\mathrm{orb}(w)$, then its attraction basin is a non–empty open set contained in the Fatou set $\mathcal{F}(T_p)$. Thus for any $z_0 \in B(\mathrm{orb}(w))$, we have $\mathrm{orb}(z_0) \subset B(\mathrm{orb}(w))$, and the sequence of iterates will never converge to a root of $p(z)$. Therefore, the following question becomes important to answer. *Is it possible for the sequence of iterates* $z_{n+1} = T_p(z_n)$ *to converge to a fixed point of* T_p *which is not a root of* p *or to an (super)attracting periodic orbit of* T_p*?*

In [**9**], B. Barna studies the behavior of Newton's method on the real line, and his classical result asserts that *if* $p(z)$ *is a polynomial of degree greater than or equal to 4, which only real roots, then* N_p *has periodic orbits of any period which are non–attractive.* He gives examples of such polynomials, as for instance the polynomial $p(z) = 3z^5 - 10z^3 + 23z$ for which $\{-1, 1\}$ is a superattracting periodic orbit for Newton's iterative function N_p. Extensions of these results are obtained in [**19**] and [**31**].

DEFINITION 2.2. (Order of convergence) *Let* $z_{n+1} = z_n - \phi(z_n)$ *be an iterative root–finding method such that for every simple root* r *of* $f(z)$ *, we have* $\phi'(r) = 1$ *,* $\phi''(r) = \cdots = \phi^{(k-1)}(r) = 0$ *and* $\phi^{(k)}(r) \neq 0$ *, then we say that the root–finding algorithm is of order at least* k *convergent.*

DEFINITION 2.3. *Let* $R_1, R_2 : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ *be two rational maps. We say that* R_1 *and* R_2 *are conjugated if there is a Möbius transformation* $\psi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ *such that* $R_2 \circ \psi(z) = \psi \circ R_1(z)$ *, for all* z *.*

An important feature of conjugation of rational maps is given by the following classical result.

THEOREM 2.1. *Let* R_1 *and* R_2 *be two rational maps, and let* ψ *be a Möbius transformation conjugating* R_1 *and* R_2 *, that is,* $R_2 = \psi \circ R_1 \circ \psi^{-1}$ *. Then* $\mathcal{F}(R_2) = \psi(\mathcal{F}(R_1))$ *and* $\mathcal{J}(R_2) = \psi(\mathcal{J}(R_1))$ *.*

From a dynamical system point of view, conjugacy plays a central role in the understanding of the behavior of classes of maps in the following sense. Suppose we wish to describe both the quantitative and the qualitative behavior of the map $z \longrightarrow T_f(z)$, where T_f is some iterative root–finding map. Since a conjugacy preserves fixed and periodic points as well as their type, and the attraction basins as well as, the dynamical data concerning f is carried by the fixed points of T_f , as well as by the nature of such fixed points which may be (super)attracting, repelling, or indifferent. Therefore, for polynomials of degree greater than or equal to two, it is worthwhile to build up parametrized families consisting of polynomials p_μ which are *as simple as possible* so that there exists a conjugacy between T_p and T_{p_μ} for a suitable choice of the complex parameter μ .

DEFINITION 2.4. (universal Julia set) *We will say that an iterative root–finding algorithm* $f \longrightarrow T_f$ *has universal Julia set for polynomials of degree* d *if there exists a rational map* R *such that for every polynomial* f *of degree* d *,* $\mathcal{J}(T_f)$ *is conjugated to* $\mathcal{J}(R)$ *by a Möbius transformation* ψ *, or in other words* $\mathcal{J}(T_f) = \psi(\mathcal{J}(R))$ *.*

The next result, which is due to A. Cayley and to E. Schröder, has great historical importance. (See [**17**], [**18**]),and [**52**].) In an attempt to understand the dynamics of Newton's method in the complex plane, they investigated the dynamics of Newton's method applied to polynomials of a particularly simple form. Cayley realized that major difficulties would arise when attempting to extend this result for quadratics to cubics and beyond. It is believed that this circumstance motivated further work of P. Fatou and G. Julia along these lines.

THEOREM 2.2. (A. Cayley [**17**], [**18**], and E. Schröder [**52**]) *Let*

$$N_f(z) = \frac{z^2 - a\,b}{2\,z - (b+a)}$$

be the rational map obtained from Newton's method applied to the quadratic polynomial $f(z) = (z-a)(z-b)$ *, with* $a \neq b$ *. Then* N_f *is conjugated to the map* $z \longrightarrow z^2$ *by the Möbius transformation* $M(z) = \frac{z-a}{z-b}$ *, and* $\mathcal{J}(N_f)$ *is the straight line in the complex plane corresponding to the locus of points equidistant from* a *and* b *.*

We have extensions of this theorem which are due to K. Kneisl. Recall that Halley's method associated to f is given by

$$H_f(z) = z - \frac{2f(z)f'(z)}{2f'(z)^2 - f(z)f''(z)} \,.$$

Theorem 2.3. (Kneisl [**35**]) *Let*

$$H_f(z) = \frac{z^3 - 3\,a\,b\,z + a\,b\,(a+b)}{3\,z^2 - 3\,(a+b)\,z + a^2 + a\,b + b^2}$$

be the rational map obtained from Halley's method applied to the quadratic polynomial $f(z) = (z-a)(z-b)$ *, with* $a \neq b$ *. Then* H_f *is conjugated to the map* $z \longrightarrow z^3$ *by the Möbius transformation* $M(z) = \frac{z-a}{z-b}$ *, and* $\mathcal{J}(H_f)$ *is the straight line in the complex plane corresponding to the locus of points equidistant from* a *and* b *.*

Recall also that Chebyshev's method, also known as the super–Newton method, associated to a map f is defined by

$$S_f(z) = z - \frac{f(z)}{f'(z)} - \frac{f(z)^2 f''(z)}{2f'(z)^3} \,.$$

Theorem 2.4. (Kneisl [**35**]) *Let*

$$S_f(z) \quad = \quad \frac{3z^4 - 2(a+b)z^3 - 6abz^2 + 6ab(a+b)z - ab(a^2+3ab+b^2)}{8\,z^3 + 12(\,a+b)\,z^2 + (a+b)^2\,z + (a+b)^3}$$

be the rational map obtained from the super–Newton method applied to the quadratic polynomial $f(z) = (z-a)(z-b)$ *, with* $a \neq b$ *. Then* S_f *is conjugated to the map* $S_3(z) = \frac{z^4+2z^3}{2z+1}$ *via the Möbius transformation* $M(z) = \frac{z-a}{z-b}$ *.*

Here we extend the results due to Schröder, Cayley, and Kneisl to the set consisting of the iterative methods under discussion.

By definition, any iterative method T_f associated to a function f has the property that the roots of f are fixed points of T_f , which are, in general, superattracting fixed points. Note that there may exist other fixed points of T_f which do not correspond to any root of f , and which we call *free fixed points*. These points can be either (super)attracting, or repelling, or indifferent. Therefore, the following three questions are natural. *Do there exist free fixed points for the iterative methods under consideration?* If so, we may ask *whether or not they are (super)attracting or indifferent fixed points.* We may also ask the following question. *Do there exist (super)attracting or indifferent periodic orbits for the iterative methods considered in this paper?*

Concerning the problem on the existence of (super)attracting periodic orbits, a general method for constructing polynomials with a (super)attracting periodic orbit of any given period for Newton's iterative method is given in [**44**]. For similar results for other iterative–root finding methods, see [**4**] as well as [**5**]. Now for other iterative root–finding methods, we may formulate the following problem. *Construct specific examples of polynomials so that when a given iterative map studied in this paper is applied to them, the resulting rational map has attracting periodic orbits of period greater than or equal to 2.*

On the other hand, if for some iterative root–finding method there are attracting fixed points which do not correspond to the roots of f or there are attracting periodic orbits, we have the following problem. *Describe attraction basins of (super)attracting fixed points corresponding to roots of polynomials, as well as to those corresponding to attracting periodic orbits.* For example, study their topological properties. *For which iterative maps is the Julia set connected? locally connected?* Concerning this problem, it was proved independently by M. Shishikura [**50**] and F. Przytycki [**46**] that the Julia set of Newton's method applied to polynomials is connected. Actually, Shishikura showed that the Julia set is connected for every rational map having only one weakly repelling fixed point, or in other words a fixed point z_0 of a rational map R such that $|R'(z_0)| > 1$ or $R'(z_0) = 1$.

As we note, for most iterative root–finding methods, the roots of f are superattracting fixed points. The critical points that do not correspond to roots of f are called *free critical points*. The reason why free critical points are important is due to the following classical result.

Theorem 2.5. (Fatou–Julia) *Let R be a rational map. Then the immediate basin of attraction of each attracting periodic point contains at least one critical point.*

Consequently, the existence of attracting periodic orbits places an obstruction to actually finding a root of f because, in such a case, the immediate basin of attraction of each attracting periodic point contains at least one critical point. Note that the search of the roots of f is not only interfered by the existence of attracting periodic orbits; in some cases there may exist additional fixed points, or in other words fixed points which are not roots of f, which may be either attracting, or repelling, or indifferent.

When we apply any of the root–finding iterative methods studied here to complex polynomials, we obtain rational maps on the Riemann sphere. In order to study affine conjugacy classes of these iterative methods, we mention the following relevant result.

Theorem 2.6. (Scaling Theorem for Newton's method, [**20**]) *Let $T(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map on the complex plane, and let $\lambda \in \mathbb{C}$ be a non–zero constant. Let $f(z)$ be a polynomial; define the polynomial $g(z) = \lambda(f \circ T)(z)$. Then Newton's methods N_f and N_g are affinely conjugated by T, that is, $N_f \circ T(z) = T \circ N_g(z)$ (scaling equation), for all z.*

In [**47**], this result is generalized for Halley's method, and in [**43**] for the families of iterative root–finding methods of König and of Schröder. Moreover, in [**16**] this theorem is also shown for König's family of iterative methods. We extend the Scaling Theorem for the remaining iterative methods considered in this paper, except for Stirling's method and that of Steffensen, that do not satisfy the scaling equation.

If the Scaling Theorem holds for the iterative methods considered here, applying these methods we reduce the study of the dynamics of the rational maps obtained.

Let R be a rational map on the Riemann sphere. The *postcritical set* of R, denoted $\mathcal{P}(R)$, is the closure of the strict forward orbits of the critical points of R. In other words, if we denote the set consisting of the critical points of R by $C(R)$, we have that

$$\mathcal{P}(R) = \overline{\bigcup_{\substack{c \in C(R) \\ n>0}} R^{\circ n}(c)}\,.$$

We have $R(\mathcal{P}(R)) \subset \mathcal{P}(R)$ and $\mathcal{P}(R^{\circ n}) = \mathcal{P}(R)$. In other words, the postcritical set is the smallest closed set containing the critical values of $R^{\circ n}$, for every $n > 0$.

A result in the line of this work is the following.

THEOREM 2.7. *(See [**37**].) Let R be a rational map on the Riemann sphere. Then the postcritical set $\mathcal{P}(R)$ of R contains the attracting periodic orbits of R, the indifferent periodic orbits which lie in the Julia set of R, the boundary of every Siegel disk (a k–periodic component U of the Fatou set of R, which is a disk on which $R^{\circ k}$ acts by an irrational rotation) or the boundary of every Herman ring (a k–periodic component U of the Fatou set of R, which is an annulus on which $R^{\circ k}$ acts by an irrational rotation).*

The importance of the description of the postcritical set of a rational map is given by the following.

THEOREM 2.8. (Characterization of hyperbolicity, see [**37**]) *Let R be a rational map. Then the following three conditions are equivalent:*

1. *The postcritical set $\mathcal{P}(R)$ is disjoint from the Julia set $\mathcal{J}(R)$.*
2. *There are neither critical points nor parabolic periodic points (an indifferent k–periodic point z_0 such that $(R^{\circ k})'(z_0)$ is a root of unity) in the Julia set.*
3. *Every critical point of R tends to an attracting periodic point under forward iterations.*

DEFINITION 2.5. *A rational map is hyperbolic if any of the three preceding equivalent conditions are satisfied.*

Concerning the size, from a measure point of view of the Julia set of a hyperbolic rational map, we have the following well known result.

THEOREM 2.9. *The Julia set of a hyperbolic rational map has zero measure.*

Remark. The measure considered in the preceding result is Lebesgue's measure on the Riemann sphere.

We now have the following problem. *For any iterative root–finding map considered in this paper, study the parameter values for which when we apply them to the quadratic family $p_c(z) = z^2 + c$, as well as to the cubic family $p_A(z) = z^3 + (A-1)z - A$, the resulting rational map is hyperbolic.*

3. Numerical Methods

In this section, we will show that the Scaling Theorem holds for all of the iterative root–finding method that we consider. Therefore, we may consider the simplest representative rational map obtained through conjugacy and attempt to describe the dynamics of the conjugacy classes. Concerning the description of parameter spaces, a study for Halley's iterative map has already been done. (See [**16**] as well as [**47**].) Furthermore, Chebyshev's iterative map appears as an element of the family of Schröder's iterative maps S_σ; indeed, it corresponds to the iterative map S_3 (we note that for $\sigma = 2$, the iterative map S_2 is Newton's iterative map). In general, some progress has been made on the study of the parameter space for cubic and quartic polynomials. (See [**22**].) Finally, for the remaining iterative maps presented here, almost nothing is known about the parameter spaces for polynomials of degree less than or equal to four.

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function. We define two maps associated to f. Let $u_f(z) = \frac{f(z)}{f'(z)}$ and let $L_f(z) = \frac{f(z)f''(z)}{f'(z)^2}$, which are analytic functions in every subregion U of $\mathbb{C}$ in which $f'(z) \neq 0$.

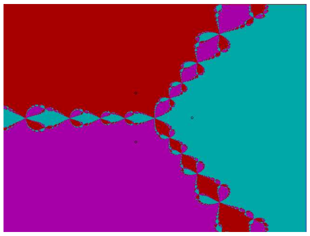
Define a new function $g(z)$ by $g = f \circ T$, where $T : \mathbb{C} \longrightarrow \mathbb{C}$ is the affine map $T(z) = \alpha z + \beta$, with $\alpha \neq 0$. Then $(g \circ T^{-1})'(z) = g'(T^{-1}(z))(T^{-1})'(z) = \frac{1}{\alpha} g'(T^{-1}(z))$, that is, $g'(T^{-1}(z)) = \alpha f'(z)$, and similarly $(g \circ T^{-1})''(z) = \frac{1}{\alpha^2} g''(T^{-1}(z))$, that is, $g''(T^{-1}(z)) = \alpha^2 f''(z)$. Proceeding similarly, we may show that $g^{(i)}(T^{-1}(z)) = \alpha^i f^{(i)}(z)$, for each $i \geqslant 1$. Now we have $u_g(T^{-1}(z)) = \frac{g(T^{-1}(z))}{g'(T^{-1}(z))} = \frac{f(z)}{\alpha f'(z)} = \frac{1}{\alpha} u_f(z)$, that is, $\alpha u_g(T^{-1}(z)) = u_f(z)$ and $L_g(T^{-1}(z)) = \frac{g(T^{-1}(z))\, g''(T^{-1}(z))}{g'(T^{-1}(z))^2} = L_f(z)$.

Note that if $\Phi(z) = z - \phi(z)$ is an arbitrary iterative root–finding map, then $T \circ \Phi \circ T^{-1}(z) = \alpha(T^{-1}(z) - \phi(T^{-1}(z))) + \beta = z - \alpha\phi(T^{-1}(z))$. Therefore, to show that an iterative method satisfies the Scaling Theorem, it suffices to prove that $\alpha\phi(T^{-1}(z)) = \phi(z)$.

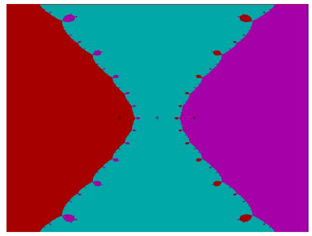
3.1. Newton's iterative map. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function. Newton's map associated to f is

$$N_f(z) = z - u_f(z) = z - \frac{f(z)}{f'(z)}.$$

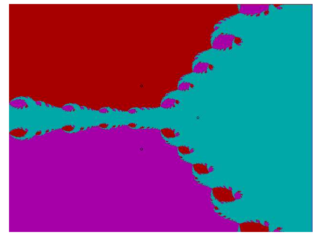
Since $N_f'(z) = \dfrac{f(z)f''(z)}{(f'(z))^2} = L_f(z)$, we have that the critical points of N_f are either roots of f or solutions of the equation $f''(z) = 0$. We have $N_f(z) = z$ if and only if $f(z) = 0$, or in other words the fixed points of N_f are the roots of f. Also, we have that $N_f(\infty) = \infty$,a s well as that $z = \infty$ is a repelling fixed point of N_f. On the other hand, if z_0 is a fixed point of N_f, then $N_f'(z_0) = \dfrac{m-1}{m}$, where m is the multiplicity of z_0 as a root of f. Thus if z_0 is a simple root of f, then z_0 is a superattracting fixed point of N_f, that is, $N_f'(z_0) = 0$. Therefore, the convergence of the iterated $N_f^n(z)$ is at least quadratic in a neighborhood of z_0. The next three pictures show the basins of attraction of the roots when we apply Newton's iterative map to the corresponding polynomials.



$f(z) = z^3 - 1$



$f(z) = z^3 - z$



$f(z) = z^3 + 0.5z - 1.5$

It is well known that Newton's iterative method satisfies the Scaling Theorem. (See [**20**].)

The parameter space for cubic polynomials is studied in [**20**]. The topology of the basins of attraction of the roots for cubic polynomials was studied by P. Roesch in her Ph. D. Thesis, which is an extension of previous work of Tan Lei. (See [**48**], as well as [**54**].)

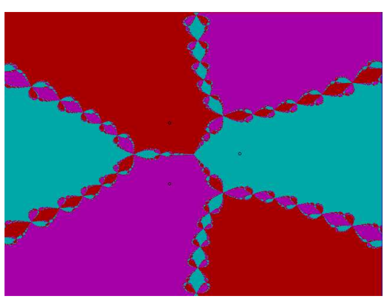
Newton's method applied to the polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, yields $N_f(z) = \dfrac{z^2 - a\,b}{2\,z - (b+a)}$, which is conjugated to the map $z \longrightarrow z^2$ via the Möbius transformation $M(z) = \frac{z-a}{z-b}$. Consequently, $\mathcal{J}(N_f)$ is the straight line in the complex plane corresponding to the locus of points equidistant from a and b.

3.2. Newton's map for multiple roots. Newton's map for multiple roots appears in a work of Schröder and is given by

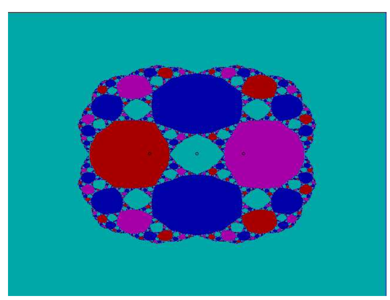
$$M_f(z) = z - \frac{f(z)f'(z)}{f'(z)^2 - f(z)f''(z)} = z - \frac{u_f(z)}{1 - L_f(z)}\,.$$

(See [**52**].) This method is an order 2 iterative map, including the case of multiple roots. It may be obtained applying Newton's iterative map to the function $u_f(z) = \dfrac{f(z)}{f'(z)}$, which has simple roots in each multiple root of f.

The next two pictures show the basins of attraction of the roots when we apply Newton's iterative map for multiple roots to the corresponding polynomials.



$f(z) = z^3 - 1$



$f(z) = z^3 - z$

For advances in the description of the parameter space for cubic polynomials see [**8**].

Schröder's method applied to the polynomial $f(z)=(z-a)(z-b)$, with $a\neq b$, yields

$$M_f(z)=\frac{(a+b)\,z^2\,-4\,a\,b\,z\,+a\,b\,(a+b)}{2\,z^2-2\,(a+b)z+a^2+b^2}\,,$$

which is conjugated to the simplest map $z\longrightarrow -z^2$ via the Möbius transformation $M_f(z)=\frac{z-a}{z-b}$. Consequently, $\mathcal{J}(M_f)$ is the straight line in the complex plane corresponding to the locus of points equidistant from a and b.

3.3. Halley's iterative map. Halley's iterative map was presented on or about 1694 by E. Halley who is well known for first computing the orbit of the comet that carries his name. This algorithm is one of the most rediscovered iterative functions of the literature. (See [**51**], [**24**] and references therein.) From its geometric interpretation for real functions, it is also known as the *method of tangent hyperbolas*. (See [**49**].) This iterative root–finding method is given by

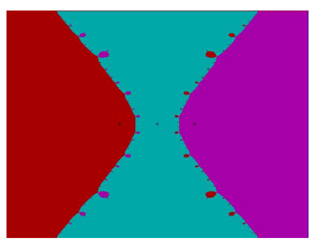
$$H_f(z)\quad=\quad z-\frac{2f(z)f'(z)}{2f'(z)^2-f(z)f''(z)}=z-\frac{2u_f(z)}{2-\,L_f(z)}\,.$$

It is well known that Halley's map is an order three iterative map in neighborhoods of simple roots.

One way to obtain this iterative map is applying Newton's method to the function $g(z)=\frac{f(z)}{\sqrt{f'(z)}}$.

The next three pictures show the basins of attraction of the roots when we apply Newton's iterative map to the corresponding polynomials.

$f(z)=z^3-1$



$f(z)=z^3-z$

$f(z)=z^3+0.5z-1.5$

Halley's method applied to the polynomial $f(z)=(z-a)(z-b)$, with $a\neq b$, yields

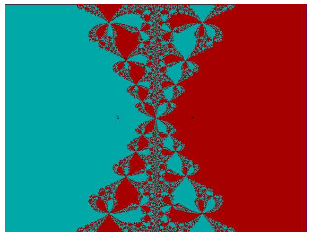
$$H_f(z)=\frac{z^3-3\,a\,b\,z\,+a\,b(a+b)}{3\,z^2-3\,(a+b)\,z+a^2+a\,b+b^2}\,,$$

which is conjugated to the map $z\longrightarrow z^3$ via the Möbius transformation $M(z)=\frac{z-a}{z-b}$. Consequently, $\mathcal{J}(H_f)$ is the straight line in the complex plane corresponding to the locus of points equidistant from a and b.

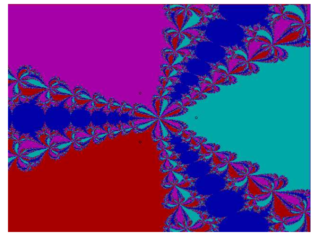
3.4. Convex acceleration of Whittaker's method. Convex acceleration of Whittaker's method is an order two iterative map given by

$$W_f(z) = z - \frac{f(z)}{2f'(z)}\,(2 - L_f(z)) = z - \frac{1}{2}\,u_f(z)(2 - L_f(z))\,.$$

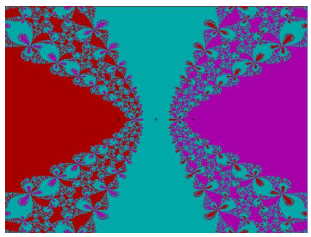
The next four pictures show the basins of attraction of the roots when we apply Whittaker's iterative map to the corresponding polynomials.



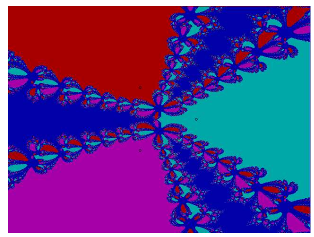
$f(z) = z^2 - 1$



$f(z) = z^3 - 1$



$f(z) = z^3 - z$



$f(z) = z^3 + 0.5z - 1.5$

From its geometrical interpretation for real functions, Whittaker's method is also known as the *parallel–chord method*. (See [**41**, p. 181].)

For the quadratic polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, Whittaker's iterative map is conjugated to the map

$$z \longrightarrow \frac{z^4 + 2z^3 + 2z^2}{2z^2 + 2z + 1}$$

via the Möbius transformation $M(z) = \frac{z-a}{z-b}$.

3.5. Double convex acceleration of Whittaker's method. The double convex acceleration of Whittaker's method is an order 3 iterative map given by

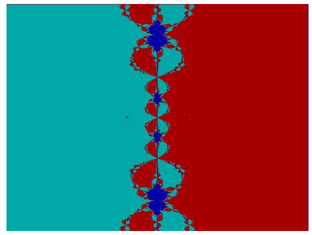
$$W_{2,f}(z) \quad = \quad z - \frac{1}{4}\,u_f(z)\left(2 - L_f(z) + \frac{4 + 2L_f(z)}{2 - L_f(z)(2 - L_f(z))}\right)\,.$$

For the quadratic polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, the iterative map $W_{2,f}$ is conjugated to the map

$$z \longrightarrow \frac{z^8 + 4z^7 + 8z^6 + 8z^5 + 4z^4}{4z^4 + 8z^3 + 8z^2 + 4z + 1}$$

via the Möbius transformation $M(z) = \frac{z-a}{z-b}$.

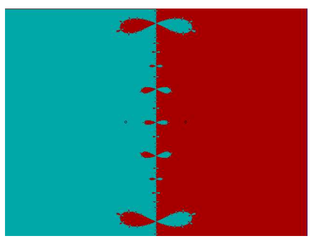
The next picture shows the basins of attraction of the roots when we apply the iterative map $W_{2,f}$ to the polynomial $f(z) = z^2 - 1$.



3.6. Chebyshev's iterative map. This iterative map is also known as the *super–Newton method*. It is an element of Schröder's family of iterative maps S_σ. (See Section 7.1.) In fact, this method corresponds to the iterative map S_3. Therefore, it is an order 3 iterative map given by

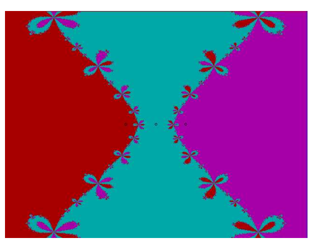
$$Cheby_f(z) \quad = \quad z - u_f(z)\left(1 + \frac{1}{2}\, L_f(z)\right) .$$

The next four pictures show the basins of attraction of the roots when we apply Chebyshev's iterative map to the corresponding polynomials.

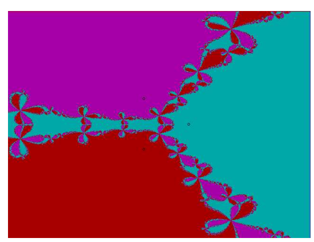


$f(z) = z^2 - 1$

$f(z) = z^3 - 1$



$f(z) = z^3 - z$



$f(z) = z^3 + 0.5z - 1.5$

Chebyshev's iterative map is also known as *Euler–Chebyshev's method*. From its interpretation for real functions, it is furthermore known as the *method of tangent parabolas*. (See [**55**].)

Chebyshev's method applied to the quadratic polynomial $f(z) = (z - a)(z - b)$, with $a \neq b$, is conjugated to the map

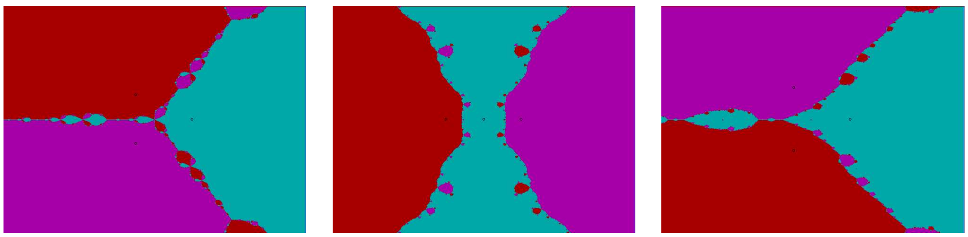
$$S_3(z) = \frac{z^4 + 2z^3}{2z + 1}$$

via the Möbius transformation $M(z) = \frac{z-a}{z-b}$.

3.7. Super–Halley iterative map. The super–Halley iterative method is also known as the *convex acceleration of Newton's method*. It is an order 3 iterative map given by

$$SH_f(z) \quad = \quad z - u_f(z)\left(1 + \frac{L_f(z)}{2(1 - L_f(z))}\right) .$$

The next three pictures show the basins of attraction of the roots when we apply super–Halley iterative map to the corresponding polynomials.



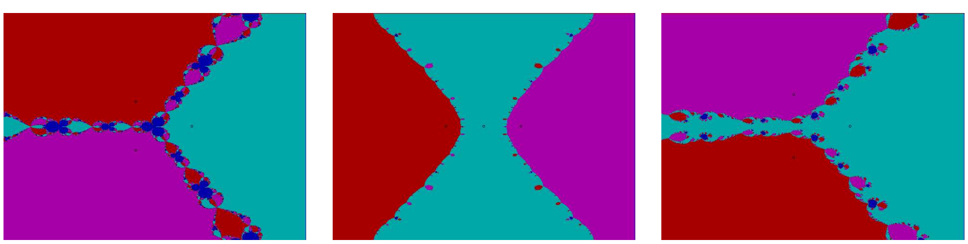
$f(z) = z^3 - 1$ $\quad$ $f(z) = z^3 - z$ $\quad$ $f(z) = z^3 + 0.5z - 1.5$

For the quadratic polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, the super–Halley iterative map is conjugated to the map $z \longrightarrow z^4$ via the Möbius transformation $M(z) = \frac{z-a}{z-b}$. Consequently, $\mathcal{J}(SH_f)$ is the straight line in the complex plane corresponding to the locus of points equidistant from a and b .

3.8. Midpoint iterative map. The midpoint iterative map is an order 3 iterative map given by

$$Mdp_f(z) = z - \frac{f(z)}{f'\left(z - \frac{f(z)}{2f'(z)}\right)} = z - \frac{f(z)}{f'\left(z - \frac{1}{2}u_f(z)\right)} .$$

The next three pictures show the basins of attraction of the roots when we apply midpoint iterative map to the corresponding polynomials.



$f(z) = z^3 - 1$ $\quad$ $f(z) = z^3 - z$ $\quad$ $f(z) = z^3 + 0.5z - 1.5$

For the quadratic polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, midpoint method Mdp_f, yields

Midpoint applied to the polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, yields the rational map

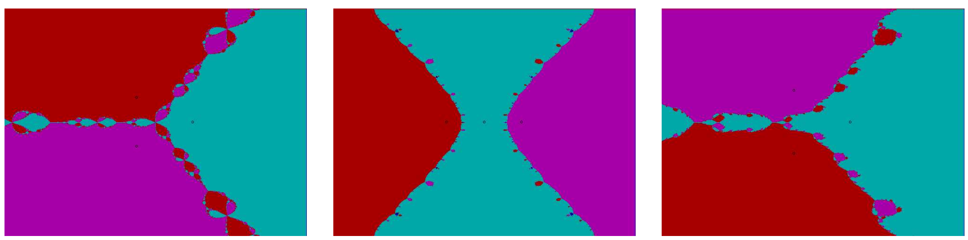
$$Mdp_f(z) = \frac{z^3 - 3\,a\,b\,z + a\,b\,(a+b)}{3z^2 - 3(a+b)z + a^2 + ab + b^2}\,,$$

which is conjugated to the simplest map $z \longrightarrow z^3$ via the Möbius transformation $M(z) = \frac{z-a}{z-b}$, . Consequently, $\mathcal{J}(Mdp_f)$ is the straight line in the complex plane corresponding to the locus of points equidistant from a and b.

3.9. Traub–Ostrowski's iterative map. Traub–Ostrowski's iterative map is an order 4 iterative map given by

$$TO_f(z) = z - u_f(z)\,\frac{f(z-u_f(z)) - f(z)}{2f(z-u_f(z)) - f(z)}\,.$$

The next three pictures show the basins of attraction of the roots when we apply Traub–Ostrowski's iterative map to the corresponding polynomials.



$f(z) = z^3 - 1$ $f(z) = z^3 - z$ $f(z) = z^3 + 0.5z - 1.5$

For the quadratic polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, Traub–Ostrowski's iterative map is given by

$$TO_f(z) = \frac{z^4 - 6abz^2 + 4ab(a+b)z - ab(a^2+ab+b^2)}{4\,z^3 - 2(a+b+2)\,z^2 + 2(a+b+a^2+b^2)\,z - (a^3 + a^2\,b + ab^2 + b^3)}\,,$$

which is conjugated to the map $z \longrightarrow z^4$ via the Möbius transformation $M(z) = \frac{z-a}{z-b}$. Consequently, $\mathcal{J}(TO_f)$ is the straight line in the complex plane corresponding to the locus of points equidistant from a and b.

3.10. Jarratt's iterative map. Jarratt's iterative map is an order 4 iterative map given by

$$J_f(z) = z - \frac{1}{2}u_f(z) + \frac{f(z)}{f'(z) - 3f'(z - \frac{2}{3}u_f(z))}\,.$$

Jarratt's method applied to the polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, yields

$$J_f(z)=\frac{z^4-6abz^2+4ab(a+b)z-ab(a^2+ab+b^2)}{4\,z^3-2(a+b+2)\,z^2+2(a+b+a^2+b^2)\,z-(a^3+a^2\,b+\,ab^2+b^3)}\,,$$

which is conjugated to the map $z\longrightarrow z^4$ via the Möbius transformation $M(z)=\frac{z-a}{z-b}$. Consequently, $\mathcal{J}(J_f)$ is the straight line in the complex plane corresponding to the locus of points equidistant from a and b.

3.11. Inverse–free Jarratt's iterative map. The inverse–free Jarratt's iterative map is an order 4 iterative map which is obtained as follows. Let $h_f(z)=\dfrac{f'\left(z-\frac{2}{3}u_f(z)\right)-f'(z)}{f'(z)}$. Then this algorithm is given by

$$IJ_f(z)\quad=\quad z-u_f(z)+\frac{3}{4}u_f(z)h_f(z)\left(1-\frac{3}{2}h_f(z)\right)\,.$$

Inverse–free Jarratt's iterative map applied to the polynomial $f(z)=(z-a)(z-b)$, with $a\neq b$, yields the rational map

$$IJ_f(z)=\frac{z^4-6abz^2+4ab(a+b)z-ab(a^2+ab+b^2)}{4\,z^3-2(a+b+2)\,z^2+2(a+b+a^2+b^2)\,z-(a^3+a^2\,b+\,ab^2+b^3)}\,,$$

which is conjugated to the map $z\longrightarrow z^4$ via the Möbius transformation $M(z)=\frac{z-a}{z-b}$. Consequently, $\mathcal{J}(IJ_f)$ is the straight line in the complex plane corresponding to the locus of points equidistant from a and b.

4. The Scaling Theorem

Concerning the conjugacy classes of the iterative root–finding maps presented in Section 3, we have the following result.

THEOREM 4.1. (Scaling Theorem) *Let $T(z)=\alpha z+\beta$, with $\alpha\neq 0$, be an affine map on the complex plane, and let $\lambda\in\mathbb{C}$ be a non–zero constant. Let $f(z)$ be a polynomial; define the polynomial $g(z)=\lambda(f\circ T)(z)$. Let $F(z)$ denote any of the iterative root–finding maps described above. Then the iterative root–finding maps F_f and F_g, which are obtained applying F to f and g, respectively, are affinely conjugated by T. In other words, $T\circ F_f\circ T^{-1}(z)=F_f(z)$ (scaling equation), for all z.*

Proof. We give the proofs for Newton's, Schröder's, and the midpoint iterative maps. For the remaining iterative root–finding maps presented in Section 3, the arguments are similar.

In the case of Newton's iterative root–finding map, we have

$$T\circ N_g\circ T^{-1}(z)\quad=\quad z-\alpha u_g(T^{-1}(z))=N_f(z)\,;$$

in the case of Schröder's iterative root–finding map, we have

$$\begin{aligned} T\circ M_g\circ T^{-1}(z) &= z-\alpha\frac{u_g(T^{-1}(z))}{1-L_g(T^{-1}(z))}\\ &= z-\frac{u_f(z)}{1-L_f(z)}\\ &= M_f(z)\,; \end{aligned}$$

and in the case of the midpoint iterative map we have

$$T\circ Mdp_g\circ T^{-1}(z) = z-\alpha\frac{g(T^{-1}(z))}{g'\left(T^{-1}(z)-\frac{1}{2}u_g(T^{-1}(z))\right)}\,.$$

Using a Taylor series, we have

$$\begin{aligned} g'(T^{-1}(z)-k\,u_g(T^{-1}(z))) &= g'(T^{-1}(z)-k\,\frac{1}{\alpha}\,u_f(z))\\ &= g'(T^{-1}(z))-g''(T^{-1}(z))\left(\frac{1}{\alpha}\,k\,u_f(z)\right)+\\ &\quad \frac{1}{2!}g'''(T^{-1}(z))\left(\frac{1}{\alpha}\,k\,u_f(z)\right)^2+\cdots\\ &= \alpha f'(z)-\alpha^2 f''(z)\left(\frac{1}{\alpha}\,ku_f(z)\right)+\\ &\quad \alpha^3\frac{1}{2!}f'''(z)\left(\frac{1}{\alpha}\,k\,u_f(z)\right)^2+\cdots\\ &= \alpha\left(f'(z)-f''(z)\,(k\,u_f(z))+\frac{1}{2!}f'''(z)\,(k\,u_f(z))^2+\cdots\right)\\ &= \alpha f'\left(z-k\,u_f(z)\right)\,. \end{aligned}$$

Setting $k=\frac{1}{2}$ in the Taylor series above, we obtain $g'(T^{-1}(z)-\frac{1}{2}u_g(T^{-1}(z)))=\alpha f'(z-\frac{1}{2}u_f(z))$. Therefore,

$$T\circ Mdp_g\circ T^{-1}(z) = z-\alpha\frac{f(z)}{\alpha f'\left(z-\frac{1}{2}\,u_f(z)\right)}=Mdp_f(z)\,,$$

which completes the proof.

5. Two Iterative root–finding maps which do not satisfy the Scaling Theorem

In this section we give two iterative root–finding maps which do not satisfy the Scaling Theorem.

5.1. Stirling's iterative map. Stirling's iterative map is an order 2 iterative map given by

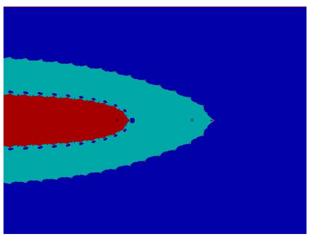
$$St_f(z) = z - \frac{f(z)}{f'(z - f(z))}\,.$$

Now we have

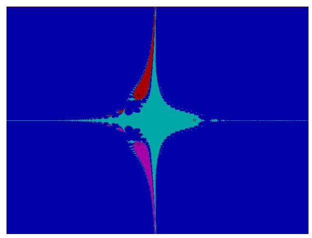
$$T \circ St_g \circ T^{-1}(z) \quad = \quad z - \alpha \frac{g(T^{-1}(z))}{g'(T^{-1}(z) - g(T^{-1}(z)))}\,,$$

and it is easy to see that this method does not satisfy the scaling equation.

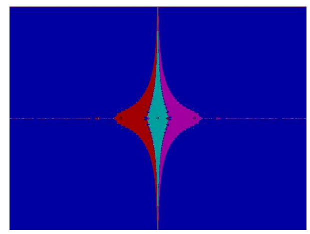
The next three pictures show the basins of attraction of the roots when we apply Stirling's iterative map to the corresponding polynomials.



$f(z) = z^2 - 1$



$f(z) = z^3 - 1$



$f(z) = z^3 - z$

For the quadratic polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, Stirling's iterative map is given by

$$St_f(z) = \frac{2z^3 - (2a + 2b + 1)z^2 + 2abz + ab}{2z^2 - 2(a + b + 1)z + a + 2ab + b}\,,$$

which is conjugated to the map

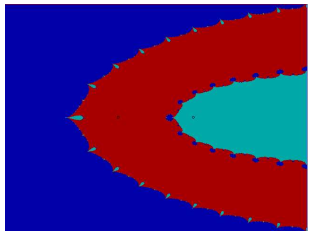
$$z \longrightarrow \frac{z^3 + (2a - 2b - 1)z^2}{(2a - 2b + 1)z - 1}$$

via the Möbius transformation $M(z) = \frac{z-a}{z-b}$.

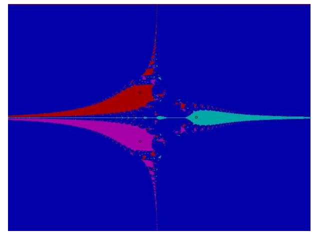
5.2. Steffensen's iterative map. Steffensen's iterative map is an order 2 iterative map obtained as follows. Defining $g_f(z) = \dfrac{f(z + f(z)) - f(z)}{f(z)}$, this algorithm is given by

$$\begin{aligned} Stef_f(z) \quad &= \quad z - \frac{f(z)}{g_f(z)} \\ &= \quad z - \frac{f(z)^2}{f(z + f(z)) - f(z)}\,. \end{aligned}$$

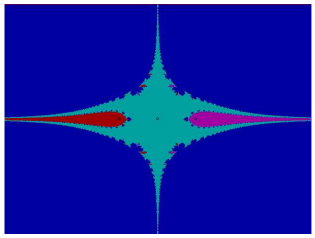
The next four pictures show the basins of attraction of the roots when we apply Steffensen's iterative map to the corresponding polynomials.



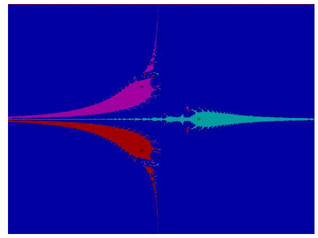
$f(z) = z^2 - 1$



$f(z) = z^3 - 1$



$f(z) = z^3 - z$



$f(z) = z^3 + 0.5z - 1.5$

As in case of Stirling's method, we see that this algorithm does not satisfy the scaling equation.

Steffensen's iterative map apply to the polynomial $f(z) = (z-a)(z-b)$, with $a \neq b$, yields the rational map

$$St_f(z) = \frac{z^3 - (a+b-1)z^2 + a\,b\,z - a\,b}{z^2 - (a+b-2)z - (a - ab + b)},$$

which is conjugated to the map

$$z \longrightarrow \frac{z^3 - (a-b+1)}{(1-a+b)z - 1}$$

via the Möbius transformation $M(z) = \frac{z-a}{z-b}$.

6. Three third–order iterative methods whichdo not require the use of second derivatives

We consider three third–order iterative root–finding methods that do not require the use of second derivatives which, for $n \geqslant 0$, are given by

$$\begin{cases} w_n = z_n - \frac{f(z_n)}{f'(z_n)}, \\ z_{n+1} = w_n - \frac{f(w_n)}{f'(z_n)}, \end{cases}$$

$$\begin{cases} w_n = z_n - \frac{f(z_n)}{f'(z_n)}, \\ z_{n+1} = z_n - \frac{2f(z_n)}{f'(z_n)+f'(w_n)}, \end{cases}$$

and

$$\begin{cases} w_n &= z_n - \frac{f(z_n)}{2f'(z_n)}, \\ \\ z_{n+1} &= z_n - \frac{f(z_n)}{f'(w_n)}. \end{cases}$$

These three third–order root–finding iterative methods are studied in [**25**], [**41**], [**45**], [**59**].

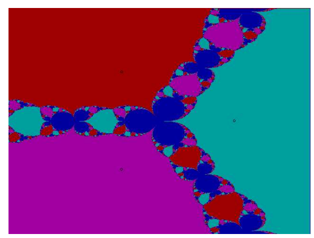
The iterative maps defining the preceding three iterative methods are given by

$$\begin{aligned} M_{1,f}(z) &= z - u_f(z) - \frac{f\left(z - u_f(z)\right)}{f'(z)}, \\ M_{2,f}(z) &= z - \frac{2f(z)}{f'(z) + f'\left(z - u_f(z)\right)}, \quad \text{and} \\ M_{3,f}(z) &= z - \frac{f(z)}{f'\left(z - \frac{1}{2}u_f(z)\right)}, \end{aligned}$$

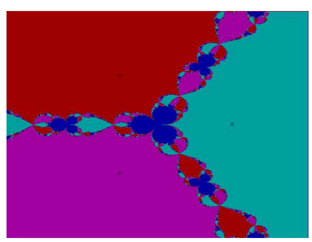
respectively.

The next three pictures show the basins of attraction of the roots when we apply iterative maps $M_{f,j}$, with $j = 1, 2, 3$, to the polynomial $f(z) = z^3 - 1$.

$M_{1,f}$–method for $f(z) = z^3 - 1$



$M_{2,f}$–method for $f(z) = z^3 - 1$



$M_{3,f}$–method for $f(z) = z^3 - 1$

Observe that when we apply the iterative maps $M_{i,f}$ (with $i = 1, 2, 3$) to a polynomial, we obtain a rational map on the Riemann sphere.

We now have the following.

Theorem 6.1. (Scaling Theorem) *Let $f(z)$ be an analytic function on the Riemann sphere, and let $T(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map. Define*

$g(z) = \lambda(f \circ T(z))$. Then $T \circ M_{i,g} \circ T^{-1} = M_{i,f}(z)$, that is, $M_{i,f}$ and $M_{i,g}$ are conjugated by T, with $i = 1, 2, 3$.

Proof. We give the proof only for the iterative map $M_{3,f}$. The arguments for the proofs of the maps $M_{j,f}$ (with $j = 1, 2$) are similar. We have

$$\begin{aligned} T \circ M_{3,g} \circ T^{-1}(z) &= z - \alpha \frac{g(T^{-1}(z)}{g'\left(T^{-1}(z) - \frac{1}{2}u_f(z)\right)} \\ & \quad z - \alpha \frac{f(z)}{\alpha f'\left(z - \frac{1}{2}u_f(z)\right)} \\ &= M_{3,f}(z). \end{aligned}$$

7. Families of iterative maps

There are many iterative maps for solving non–linear equations which depend on one or more complex parameter(s). In this case, we speak of *families of iterative root–finding maps.*

7.1. Schröder's family of iterative maps. Let f be an analytic function on the Riemann sphere, and let σ be a positive integer greater than or equal to two. For each σ, with $\sigma = 1, 2, 3, \ldots$, Schröder's iterative map $S_{\sigma,f}$ is an algorithm of order σ.

We next show how to obtain Schröder's iterative maps. Let F be an analytic function, and let $|h|$ be sufficiently small. Then by the Taylor expansion

$$F(z+h) = F(z) + \sum_{n=1}^{\infty} b_n(z) h^n \; = \; F(z) + B(z)$$

where $b_n(z) = \frac{F^{(n)}(z)}{n!}$, with $n = 1, 2, \ldots$, the function $B(z)$ may be considered as a formal power series in the variable h whose coefficients depend on z. If $b_1(z) = F'(z) \neq 0$, then $B(z)$ has a formal inverse $B^{-1}(z)$. Thus we may write

$$B^{-1}(z) = \sum_{n=1}^{\infty} c_n(z)\, h^n \,,$$

where $c_n(z) = \frac{1}{n}\, res(B^{-n})$, with $n = 1, 2, \ldots$, by the Lagrange–Bürmann formula.

Using the approximation above for an analytic function, we may define Schröder's iterative maps of order $\sigma = 2, 3, \ldots$ associated to f by

$$S_{\sigma,f}(z) = z + \sum_{k=1}^{\sigma-1} c_k(z)\, (-f(z))^k \,,$$

where

$$c_n(z) = \frac{1}{k!}\left(\frac{1}{f'(z)}\frac{d}{dz}\right)^{(k-1)}\frac{1}{f'(z)}$$

and

$$\left(\frac{1}{f'(z)}\frac{d}{dz}\right)^{(k-1)} = \underbrace{\left(\frac{1}{f'(z)}\frac{d}{dz}\left(\frac{1}{f'(z)}\frac{d}{dz}\left(\cdots\left(\frac{1}{f'(z)}\frac{d}{dz}\right)\cdots\right)\right)\right)}_{(k-1)-\,factors}.$$

The coefficients $c_n(z)$ are analytic functions in every region $U \subset \mathbb{C}$ in which $f'(z) \neq 0$.

For example, Schröder's iterative maps $S_{3,f}$ and $S_{4,f}$ associated to f are given by

$$S_{3,f}(z) = z - \frac{f(z)}{f'(z)} - \frac{f''(z)}{2f'(z)^3}f(z)^2 = z - u_f(z)\left(1 + \frac{1}{2}L_f(z)\right),$$

which is Chebyshev's method, and by

$$S_{4,f}(z) = z - \frac{f(z)}{f'(z)} - \frac{f''(z)}{2f'(z)^3}f(z)^2 - \frac{\left(3f''(z)^2 - f'(z)f'''(z)\right)}{6f'(z)^5}f(z)^3,$$

respectively. Note that $S_{2,f} = N_f$, or in other words Schröder's iterative map of order 2 corresponds to Newton's iterative map.

In order to obtain a more explicit formula for $S_{\sigma,f}$, set $h_{1,f}(z) = 1$ and $h_{k+1,f}(z) = h'_{k,f}(z)f'(z) - (2k-1)h_{k,f}(z)f''(z)$, with $k = 1, 2, \ldots$. It then follows that

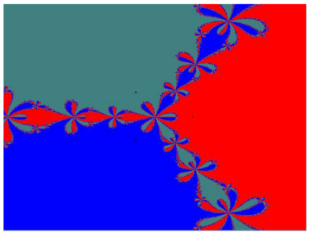
$$\frac{1}{f'(z)}\left(\frac{h_{k,f}(z)}{f'(z)^{2k-1}}\right)' = \frac{h_{k+1,f}(z)}{f'(z)^{2k+1}}.$$

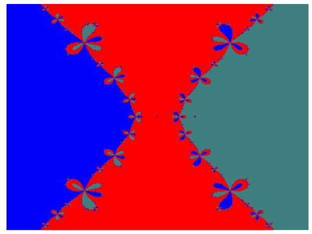
Thus the formula for $S_{\sigma,f}$ becomes

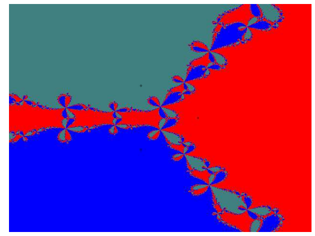
$$S_{\sigma,f}(z) = z + \sum_{k=1}^{\sigma-1}\frac{(-1)^k}{k!}\frac{h_{f,k}(z)}{(f'(z))^{2k-1}}(f(z))^k.$$

For a proof that Schröder's family of iterative methods satisfies the Scaling Theorem, see [**43**].

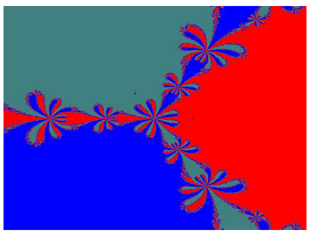
The next three pictures show the basins of attraction of the roots when we apply the iterative map $S_{3,f}$ to the corresponding polynomials,

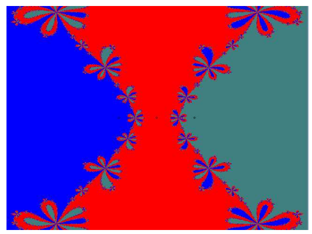

$f(z) = z^3 - 1$

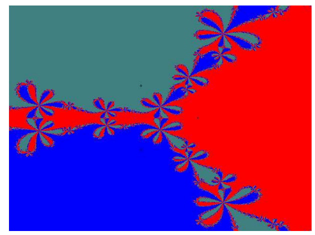

$f(z) = z^3 - z$


$f(z) = z^3 + 0.5z - 1.5$

and the next three pictures show the basins of attraction of the roots when we apply the iterative map $S_{4,f}$ to the corresponding polynomials,


$f(z) = z^3 - 1$


$f(z) = z^3 - z$


$f(z) = z^3 + 0.5z - 1.5$

7.2. König's familiy of iterative maps. Consider König's family of iterative maps. For an analytic function f on the Riemann sphere and a positive integer σ greater than or equal to two, König's iterative map of order σ, denoted $K_{\sigma,f}$, associated to f is defined by

$$K_{\sigma,f}(z) = z + (\sigma-1)\,\frac{\left(\frac{1}{f(z)}\right)^{(\sigma-2)}}{\left(\frac{1}{f(z)}\right)^{(\sigma-1)}}\,.$$

Note that $K_{2,f} = N_f$ (Newton's map) and that $K_{3,f} = H_f$ (Halley's map). For $\sigma = 4$ we have

$$K_{4,f}(z) = z - \frac{3f(z)\left(f(z)f''(z) - 2f'(z)^2\right)}{6f(z)f'(z)f''(z) - 6f'(z)^3 - f(z)^2 f'''(z)}\,.$$

It is clear that the construction of $K_{\sigma,f}$ requires the computation of the first $\sigma-1$ derivatives of f. Letting $h_{1,f}(z) = 1$ and $h_{k+1,f}(z) = h'_{k,f}(z)f(z) - kh_{k,f}(z)f'(z)$, for $k = 1, 2, \ldots, \sigma-1$, we have

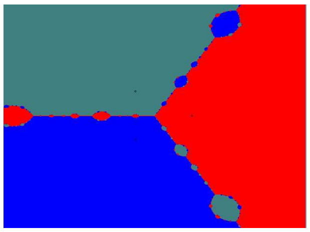
$$\left(\frac{1}{f(z)}\right)^{(k)} = \frac{h_{k+1,f}(z)}{f(z)^{k+1}}\,.$$

Thus we may write

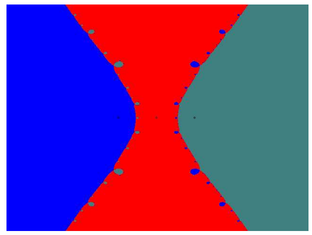
$$K_{\sigma,f}(z) = z + (\sigma - 1)\,\frac{h_{\sigma-1}(z)f(z)}{h_\sigma(z)}\,.$$

For a proof that König family of iterative methods satisfies the Scaling Theorem, see [**43**], as well as [**16**].

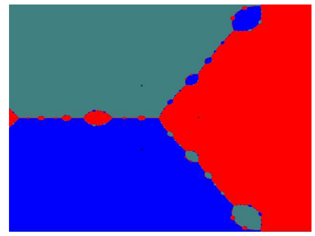
The next three pictures show the basins of attraction of the roots when we apply the iterative map $K_{4,f}$ to the corresponding polynomials.



$f(z) = z^3 - 1$



$f(z) = z^3 - z$



$f(z) = z^3 + 0.5z - 1.5$

7.3. A family of third–order iterative methods. We consider the family of third–order iterative root–finding methods

$$z_{n+1} = M_{f,\theta,c}(z_n) = z_n - \left(1 + \frac{L_f(z_n)}{2\,(1 - \theta L_f(z_n))} + cL_f(z_n)^2\right) u_f(z_n)\,,$$

where z_0 is an initial guess, and θ and c are complex parameters both to be chosen conveniently for each case. This family of iterative root–finding methods is induced by the family of iterative maps

$$M_{f,\theta,c}(z) = z - \left(1 + \frac{L_f(z)}{2\,(1 - \theta L_f(z))} + cL_f(z)^2\right) u_f(z)\,.$$

Observe that when we apply the iterative maps $M_{f,\theta,c}$ to a polynomial, we obtain a rational map on the Riemann sphere.

As particular cases, this family of third–order iterative methods includes the following (see [**3**]).

(1) When $c = 0$ and the parameter θ is real and non–negative, we obtain the family of third-order iterative functions $M_{f,\theta}(z) = M_{f,\theta,0}(z)$ studied in [**28**], which is called *Chebyshev–Halley's family of iterative root–finding maps*

$$CH_{f,\theta} = M_{f,\theta}(z) = z - \left(1 + \frac{L_f(z)}{2(1 - \theta L_f(z))}\right) u_f(z)\,.$$

In particular, we have

(a) The well known iterative function Ch_f due to Chebyshev

$$Ch_f(z) = z - \left(1 + \frac{1}{2}\,L_f(z)\right) u_f(z)$$

is obtained from the family above when $\theta = 0$.

(b) The well known iterative function H_f due to Halley.

$$H_f(z) = z - \frac{2f(z)f'(z)}{2f'(z)^2 - f(z)f''(z)}$$

is obtained from the family above when $\theta = \frac{1}{2}$. The dynamics of Halley's iterative function is studied in [**16**] and [**47**].

(c) Another third–order iterative function is the *super–Halley iterative function*, denoted SH_f, which is given by

$$SH_f(z) = z - \left(1 + \frac{L_f(z)}{2(1 - L_f(z))}\right) u_f(z),$$

and is obtained from the family above when $\theta = 1$. (See [**29**].)

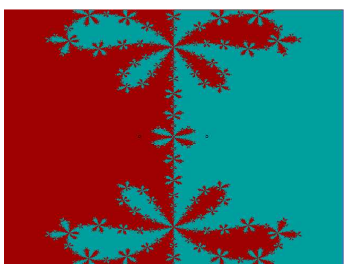
(d) Finally, Newton's method is obtained as the limit case when $\theta \longrightarrow \pm\infty$.

(2) When c is a non–zero constant and $\theta = 0$, we obtain the so–called *c–iterative methods*

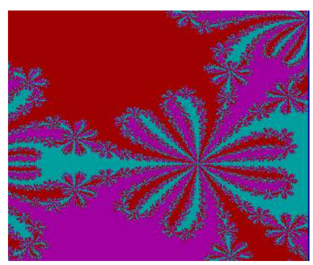
$$M_{f,c}(z) = z - \left(1 + \frac{1}{2}L_f(z) + cL_f(z)^2\right) u_f(z),$$

in which case $M_{f,c} = M_{f,0,c}$. This family of iterative map is introduced in [**2**] and [**24**].

The next two pictures show the basins of attraction of the roots when we apply c–iterative method to the corresponding polynomials.



c–method for $z^2 - 1$



c–method for $z^3 - 1$

We now show that Chebyshev–Halley's family of iterative root–finding maps satisfies the Scaling Theorem.

Theorem 7.1. (Scaling Theorem for Chebyshev–Halley's family) *Let $T(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map on the complex plane, and let $\lambda \in \mathbb{C}$ be a non–zero constant. Let $f(z)$ be polynomial; define the polynomial $g(z) = \lambda(f \circ T)(z)$. Then $M_{f,\theta,c}$ and $M_{g,\theta,c}$ are affinely conjugated by T, that is, $T \circ M_{g,\theta,c} \circ T^{-1}(z) = M_{f,\theta,c}(z)$, for all z.*

Proof. We have

$$\begin{aligned} T\circ M_{g,\theta,c}\circ T^{-1}(z) &= z-\alpha u_g(T^{-1}(z))\left(1+\frac{L_g(T^{-1}(z))}{2(1-\theta L_g(T^{-1}(z)))}+cL_g(T^{-1}(z))^2\right) \\ &= z-u_f(z)\left(1+\frac{L_f(z)}{2(1-\theta L_f(z))}+cL_f(z)^2\right) \\ &= M_{f,\theta,c}(z)\,. \end{aligned}$$

Therefore, the family of iterative root–finding maps $M_{f,\theta,c}$ satisfies the Scaling Theorem.

For the quadratic polynomial $f(z)=(z-a)(z-b)$, with $a\neq b$, we have that $CH_{f,\theta}(z)$ is conjugated to the map

$$z\longrightarrow\frac{z^4+2(1-\theta)z}{2(1-\theta)z+1}$$

via the Möbius transformation $M(z)=\frac{z-a}{z-b}$.

7.4. A King family of multipoint iterative methods. The following order four multipoint family of iterative root–finding methods is studied by R. King in [**34**].

$$K_{\beta,f}(z)=z-u_f(z)-\frac{f\left(z-u_f(z)\right)}{f'(z)}\cdot\frac{f(z)+\beta f\left(z-u_f(z)\right)}{f(z)+(\beta-2)f\left(z-u_f(z)\right)}$$

where β is a complex parameter. The family $K_{\beta,f}$ contains Traub–Ostrowski's iterative map, which is obtained when $\beta=0$. In other words,

$$\begin{aligned} TO_f(z)=K_{0,f}(z) &= z-u_f(z)-\frac{f(z-u_f(z))}{f'(z)}\,\frac{f(z)}{f(z)-2f(z-u_f(z))} \\ &= z-u_f(z)\,\frac{f(z-u_f(z))-f(z)}{2f(z-u_f(z))-f(z)}\,. \end{aligned}$$

Theorem 7.2. (Scaling Theorem) *Let $T(z)=\alpha z+\beta$, with $\alpha\neq 0$, be an affine map on the complex plane, and let $\lambda\in\mathbb{C}$ be a non–zero constant. Let $f(z)$ be a polynomial; define the polynomial $g(z)=\lambda(f\circ T)(z)$. Then $K_{\beta,f}$ and $K_{\beta,g}$ are affinely conjugated by T, or in other words $T\circ K_{\beta,g}\circ T^{-1}(z)=K_{\beta,f}(z)$, for all z.*

Proof. We have

$$\begin{aligned} T\circ K_{\beta,g}\circ T^{-1}(z) &= z-\alpha u_g(T^{-1}(z))-\alpha\frac{g\left(T^{-1}(z)-u_g(T^{-1}(z))\right)}{g'(T^{-1}(z))}\cdot \\ &\quad \frac{g(T^{-1}(z))+\beta g\left(T^{-1}(z)-u_g(T^{-1}(z))\right)}{g(T^{-1}(z))+(\beta-2)g\left(T^{-1}(z)-u_g(T^{-1}(z))\right)} \\ &= z-u_f(z)-\alpha\frac{f(z-u_f(z))}{\alpha f'(z)}\cdot\frac{f(z)+f(z-u_f(z))}{f(z)-f(z-u_f(z))} \\ &= K_{\beta,f}(z)\,. \end{aligned}$$

Therefore, the family of iterative root–finding maps $K_{\beta,f}$ satisfies the Scaling Theorem.

For the quadratic polynomial $f(z)=(z-a)(z-b)$, with $a\neq b$, we have that $K_{\beta,f}$ is conjugated to the one–parameter family of rational maps

$$z\longrightarrow\frac{z^6+(\beta+2)z^5+(1+2\beta)z^4}{(1+2\beta)z^2+(\beta+2)z+1}$$

via the Möbius transformation $M(z)=\frac{z-a}{z-b}$.

7.5. Another King family of multipoint iterative methods. Let δ, a_1, a_2 and a_3 be complex parameters. Define the following family of iterative root–finding maps

$$\tilde{K}_{\delta,a_1,a_2,a_3,f}(z)=z-a_1u_f(z)-a_2\,\frac{f(z-\delta u_f(z))}{f'(z)}-a_3\,\frac{\left(\frac{f(z-\delta u_f(z))}{f'(z)}\right)^2}{u_f(z)}\,.$$

For $\delta=1$, $a_1=1$, $a_2=1$ and $a_3=2$, we obtain the following root–finding iterative method

$$\tilde{K}_f(z)=z-u_f(z)-\frac{f(z-u_f(z))}{f'(z)}\,\frac{f(z)+2f(z-u_f(z))}{f(z)}\,.$$

THEOREM 7.3. *(Scaling Theorem) Let $T(z)=\alpha z+\beta$, with $\alpha\neq 0$, be an affine map on the complex plane, and let $\lambda\in\mathbb{C}$ be a non–zero constant. Let $f(z)$ be a polynomial; define the polynomial $g(z)=\lambda(f\circ T)(z)$. Then $\tilde{K}_{\delta,a_1,a_2,a_3,f}$ and $\tilde{K}_{\delta,a_1,a_2,a_3,g}$ are affinely conjugated by T, that is, $T\circ\tilde{K}_{\delta,a_1,a_2,a_3,g}\circ T^{-1}(z)=\tilde{K}_{\delta,a_1,a_2,a_3,f}(z)$, for all z.*

Proof. We have

$$
\begin{aligned}
T \circ \tilde{K}_{\delta,a_1,a_2,a_3,g} \circ T^{-1}(z) &= z - \alpha a_1 u_g(T^{-1}(z)) - \alpha a_2 \, \frac{g(T^{-1}(z) - \delta u_g(T^{-1}(z)))}{g'(T^{-1}(z))} - \\
&\quad \alpha a_3 \, \frac{\left(\frac{g(T^{-1}(z) - \delta u_g(T^{-1}(z)))}{g'(T^{-1}(z))}\right)^2}{u_g(T^{-1}(z))} \\
&= z - a_1 u_f(z) - \alpha a_2 \frac{f(z - \delta u_f(z))}{\alpha f'(z)} - \delta a_3 \frac{\left(\frac{f(z-\delta u_f(z))}{\alpha f'(z)}\right)^2}{\frac{1}{\alpha} u_f(z)} \\
&= \tilde{K}_{\delta,a_1,a_2,a_3,f}(z) \, ,
\end{aligned}
$$

which proves that the family of iterative root–finding maps $\tilde{K}_{\delta,a_1,a_2,a_3,f}$ satisfies the Scaling Theorem.

7.6. A Jarratt family of iterative root–finding methods. Any method (a member) contained in the Jarratt family is an order four iterative root–finding method.

Let $h_f(z) = \frac{f'\left(z - \frac{2}{3} u_f(z)\right) - f'(z)}{f'(z)}$, and let β be a complex parameter. Define a the Jarratt family of iterative root–finding maps by

$$
\Psi_{f,\beta}(z) = z - u_f(z) + \frac{3}{4} \, u_f(z) h_f(z) \, \frac{1 + \beta h_f(z)}{1 + \left(\frac{3}{2} + \beta\right) h_f(z)} \, .
$$

This family contains Jarratt's iterative map, which is obtained by setting $\beta = 0$. For $\beta = -\frac{3}{2}$, we obtain the inverse–free Jarratt's map.

Theorem 7.4. *Let $T(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map on the complex plane, and let $\lambda \in \mathbb{C}$ be a non–zero constant. Let $f(z)$ be polynomial; define the polynomial $g(z) = \lambda (f \circ T)(z)$. Then the Jarratt families of iterative root–finding methods $\Psi_{f,\beta}$ and $\Psi_{g,\beta}$ are affinely conjugated by T, that is, $T \circ \Psi_{g,\beta} \circ T^{-1}(z) = \Psi_{f,\beta}(z)$, for all z.*

Proof. We have

$$
\begin{aligned}
T \circ \Psi_{g,\beta} \circ T^{-1}(z) &= z - \alpha u_g(T^{-1}(z)) + \\
&\quad \alpha \frac{3}{4} \, u_g(T^{-1}(z)) h_g(T^{-1}(z)) \, \frac{1 + \beta h_g(T^{-1}(z))}{1 + \left(\frac{3}{2} + \beta\right) h_g(T^{-1}(z))} \\
&= z - u_f(z) + \frac{3}{4} \, u_f(z) h_f(z) \, \frac{1 + \beta h_f(z)}{1 + \left(\frac{3}{2} + \beta\right) h_f(z)} \\
&= \Psi_{f,\beta}(z) \, ,
\end{aligned}
$$

and hence the family $\Psi_{f,\beta}$ satisfies the Scaling Theorem.

For the quadratic polynomial $f(z)=(z-a)(z-b)$, with $a\neq b$, the iterative map $\Psi_{f,\beta}(z)$ is conjugated to the map

$$z\longrightarrow\frac{3z^6+(6-4\beta)z^5+(3-8\beta)z^4}{(3-8\beta)z^2+(6-4\beta)z+3}$$

via the Möbius transformation $M(z)=\frac{z-a}{z-b}$.

7.7. A Steffensen family of multipoint iterative root–finding maps. Define $g_{\beta,f}(z)=\dfrac{f(z+\beta f(z))-f(z)}{\beta f(z)}$, where β is a complex parameter. Define the multipoint family of iterative root–finding maps $\Lambda_f(z)$ by

$$\Lambda_{\beta,f}(z)=z-\frac{f(z)}{g_f(z)}=z-\frac{\beta f(z)^2}{f(z+\beta f(z))-f(z)}.$$

Any method (a member) contained in this family is an order 2 iterative root–finding method. This family contains, as a particular case, Steffesen's map, which is obtained by setting $\beta=1$. Note that this family of iterative methods does not satisfy the Scaling Theorem.

For the quadratic polynomial $f(z)=(z-a)(z-b)$, with $a\neq b$, the iterative map $\Lambda_f(z)$ is given by

$$\Lambda_{\beta,f}(z)=\frac{\beta z^3-(a\beta+b\beta-1)z^2+ab\beta z-ab}{\beta z^2-(a\beta+b\beta-2)z+ab\beta-a-b},$$

which is conjugated to the map

$$z\longrightarrow\frac{z^3-(a\beta-b\beta+1)z^2}{(a\beta-b\beta-1)z+1}$$

via the Möbius transformation $M(z)=\frac{z-a}{z-b}$.

7.8. A Murakami family of iterative root–finding maps. Let

$$\begin{aligned}
w_2(z) &= \frac{f(z)}{f'(z-u_f(z))},\\
w_3(z) &= \frac{f(z)}{f'(z+\beta u_f(z)+\gamma w_2(z))},\\
\psi(z) &= \frac{f(z)}{b_1f'(z)+b_2f'(z-u_f(z))},
\end{aligned}$$

where β, γ, b_1 and b_2 are complex parameters. We define a Murakami family of the iterative root–finding maps by

$$Mk_f(z)=z-a_1u_f(z)-a_2w_2(z)-a_3w_3(z)-\psi(z),$$

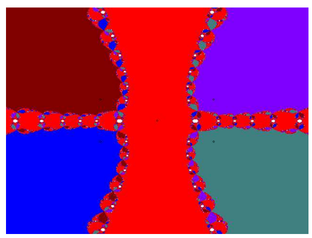
where a_1, a_2 and a_3 are complex parameters.

A proof that the Scaling Theorem holds for this family of iterative root–finding maps is obtained through a great deal of calculations

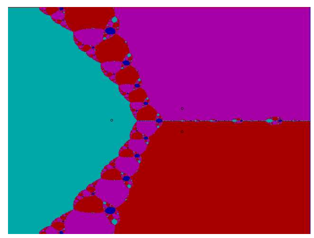
8. Constructing attracting periodic orbits for three third–order iterative methods

The problem of the existence of periodic orbits for Newton's iterative map associated to real polynomials was first studied by B. Barna. (See [**9**].) For other contributions to this problem, see [**31**]. In [**9**], we find the well known examples given by the polynomials $p(z) = 3z^5 - 10z^3 + 23z$ and $p(z) = 11z^6 - 34z^4 + 39z^2$ whose Newton maps have $\{-1, 1\}$ as a superattracting periodic orbit of period two. Another example of such a polynomial is given by $f(z) = z^3 - z + \frac{\sqrt{2}}{2}$.

The next two picture show the basin of attraction of the roots when we apply Newton's iterative map to the corresponding polynomials.



$p(z) = 3z^5 - 10z^3 + 23z$



$p(z) = z^2 - z + \frac{\sqrt{2}}{2}$

In [**44**], a general method for constructing polynomials whose Newton maps have a superattracting periodic orbit of period $n \geqslant 2$ is presented.

Now we are interested in constructing a periodic orbit for the iterative methods $M_{k,f}$ given in Section 6.

For a proof of the following results, see [**6**].

Proposition 8.1. *Let* $\Omega = \{x_1, x_2, \dots, x_n\}$ *be a set of* n *distinct complex numbers, and let* f *be a complex analytic function. Then* Ω *is a periodic orbit of period* n *of iteration maps* $M_{k,f}$ *(with* $k = 1, 2, 3$ *) if and only if*

$$(8.1) \quad \begin{cases} f'(x_i) = \dfrac{f(x_i) + f(x_i - u_f(x_i))}{x_i - x_{i+1}}, & \text{with } i = 1, \dots, n-1, \\ f'(x_n) = \dfrac{f(x_n) + f(x_n - u_f(x_n))}{x_n - x_1} \end{cases}$$

holds for $M_{1,f}$ *;*

$$(8.2) \quad \begin{cases} f'(x_i) + f'(x_i - u_f(x_i)) = \dfrac{2f(x_i)}{x_i - x_{i+1}}, & \text{with } i = 1, \dots, n-1, \\ f'(x_n) + f'(x_n - u_f(x_n)) = \dfrac{2f(x_n)}{x_n - x_1} \end{cases}$$

holds for $M_{2,f}$ *;*

$$(8.3)\qquad \begin{cases} f'\left(x_i - \frac{1}{2}\,u_f(x_i)\right) &= \dfrac{f(x_i)}{x_i - x_{i+1}} \qquad \textit{with } i = 1, \dots, n-1\,, \\ f'\left(x_n - \frac{1}{2}\,u_f(x_n)\right) &= \dfrac{f(x_n)}{x_n - x_1} \end{cases}$$

holds for $M_{3,f}$.

Proposition 8.2. *For any positive integer* $n \geqslant 2$ *and* $k = 1, 2, 3$ *, there exists a polynomial* f_k *of degree less than or equal to* $3n-1$ *for which* M_{k,f_k} *has a periodic orbit of period* n *.*

Example 8.1. Let us give a polynomial f for which the iterative method $M_{1,f}$ has a periodic orbit of period 2. For this, let $x_1 = 0$, and let $x_2 = 2$. We must construct a polynomial f such that $M_{1,f}(0) = 2$ and $M_{1,f}(2) = 0$. The polynomial f is given by $f(x) = 1 + x - \dfrac{31}{18}x^2 + \dfrac{17}{18}x^3 - \dfrac{5}{18}x^4 + \dfrac{1}{18}x^5$. A computation yields $M_{1,f}(0) = 2$ and $M_{1,f}(2) = 0$, that is, $\Omega = \{0, 2\}$ is a periodic orbit of period 2 for the method $M_{1,f}(x)$.

Example 8.2. Let us give a polynomial f for which the iterative method $M_{2,f}$ has a periodic orbit of period 2. For this, we consider $\Omega = \{x_1, x_2\}$, where $x_1 = 0$ and $x_2 = 2$. The polynomial is given by $f(x) = 1 - \frac{1}{2}x - \frac{3}{4}x^2 + \frac{1}{4}x^3$. We have that $M_{2,f}(0) = 2$ and $M_{2,f}(2) = 0$, that is, Ω is a periodic orbit of period 2 for $M_{2,f}$.

Proposition 8.3. *Let* f *be a polynomial for which* $M_{k,f}$ *(with* $k = 1, 2, 3$ *) has a periodic orbit of period* n *, say* $\Omega = \{x_1, x_2, \dots, x_n\}$ *. If* $f''(x_i) = 0$ *, for some* $i = 1, \dots, n$ *, then* Ω *is a superattracting periodic orbit of period* n *for* $M_{1,f}$ *. If* $f''(x_i) = 0$ *and* $f'(x_i) = f'(x_i - u_f(x_i))$ *for some* $i = 1, \dots, n$ *, then* Ω *is a superattracting periodic orbit of period* n *for* $M_{2,f}$ *. Finally, if* $f''(x_i - \frac{1}{2}u_f(x_i)) = 0$ *and* $f'(x_i) = f'(x_i - \frac{1}{2}u_f(x_i))$ *for some* $i = 1, \dots, n$ *, then* Ω *is a superattracting periodic orbit of period* n *for* $M_{3,f}$ *.*

The following gives a procedure for constructing a superattracting periodic orbit for the iterative methods $M_{i,f}$, with $i = 1, 2, 3$.

Theorem 8.1. *Let* $n \geqslant 2$ *be an integer. Then there exist polynomials* $\tilde{f}_k$ *(with* $k = 1, 2, 3$*) of degree less than or equal to* $3n$ *(resp.* $3n-1$ *) such that when we apply the iterative methods* $M_{k,f}$ *, with* $k = 1, 3$ *, (resp.* $M_{2,f}$ *) to* $\tilde{f}_k$ *, we have a superattracting periodic orbit of period* n *for the corresponding iterative methods.*

Example 8.3. . We continue with Example 8.1. For $M_{1,f}$, we consider $\tilde{f}(x) = f(x) + a_7 f_7(x)$, where f is the polynomial given in Example 8.1 and a_7 is a parameter to be determined. The polynomial obtained in this case is $\tilde{f}(x) = 1 + x - \dfrac{7}{9}x^3 - \dfrac{113}{72}x^4 + \dfrac{16}{9}x^5 - \dfrac{31}{72}x^6$. Now $M_{1,\tilde{f}}(0) = 2$ and $M_{1,\tilde{f}}(2) = 0$, or in other words, $\Omega = \{0, 2\}$ is a periodic orbit of period 2 for $M_{1,\tilde{f}}$. On the other hand, an easy calculation shows that $(M^2_{1,\tilde{f}})'(0) = 0$. Therefore, Ω is a superattracting periodic orbit of period 2 for $M_{1,\tilde{f}}$.

Example 8.4. For Example 8.2 and in order to construct a superattracting periodic orbit for $M_{2,f}$, we consider the polynomial $\tilde{f}(x) = f(x) + a_5 f_5(x)$ where f is the polynomial given in Example 8.2 and a_5 is a parameter to be determined. Using the condition of Theorem 8.1, we have $\tilde{f}(x) = f(x) + \frac{3}{64}x^2(x-2)^2$. We have that $M_{2,\tilde{f}}(0) = 2$ and that $M_{2,\tilde{f}}(2) = 0$, that is, $\Omega = \{0, 2\}$ is a periodic orbit of period 2 for $M_{2,\tilde{f}}$. We finally have that Ω is a superattracting periodic orbit of period 2 for $M_{2,\tilde{f}}$.

Since the hyperbolic periodic orbits of a rational map are stable, we have the following.

Theorem 8.2. *There is an open set $\mathcal{U}$ of the set consisting of the analytic complex functions such that for any $f \in \mathcal{U}$, the associated iterative method $M_{k,f}$ (with $k = 1, 2, 3$) has an attracting periodic orbit of period greater than or equal to two.*

Remark . For King's and Jarratt's iterative maps we have similar results. (See [**5**].) We show a similar result for the family (7.3). (See [**4**].)

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Amat and Busquier:
Departamento de Matemática Aplicada y Estadística
Universidad Politécnica de Cartagena
SPAIN
E-mail address: sergio.amat@upct.es
E-mail address: sonia.busquier@upct.es

Plaza:
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Santiago de Chile
Casilla 307, Correo 2. Santiago
CHILE
E-mail address: splaza@lauca.usach.cl