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On distortion under hyperbolically convex maps

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Abstract. We study the class of hyperbolically convex bounded univalent functions with a boundary normalization in the unit disk U. In the paper we obtain the lower estimate for the distortion in this class. A two-point distortion theorem is also proved. The method of proofs is based on the reduced modulus of digons and the modulus of annuli.

1. Introduction

We study the class H of all holomorphic and univalent functions $f(z)$ in the unit disk $U = \{z : |z| < 1\}$ such that $f(0) = 0$, $f'(0) > 0$, and $f(U)$ is hyperbolically convex (*h-convex*) in U. *h*-convexity means that each segment of the hyperbolic plane U connecting two points of $f(U)$ lies in $f(U)$. Such domains are of great importance, in particular, in the theory of Fuchsian groups [1] where the normal fundamental domain of a Fuchsian group is h-convex.

An important property of h-convex functions is the invariance under the group $\text{Möb}(U)$ of conformal automorphisms of the unit disk. Namely, if f is h-convex, then the mapping

$$\sigma \circ f \circ \tau \quad \text{with } \sigma, \tau \in \text{Möb}(U)$$

is also h-convex. Therefore, it is always possible to achieve the normalization given by the following boundary condition. Let us denote by H_c the class of all functions from H with $c = \inf_{w \in U \setminus f(U)} |w|$ fixed. Roughly speaking, we fix the minimal distance between the origin and the boundary of $f(U)$. We are aimed at the application of the extremal length method to solve extremal problems in this class. The class H_c has been introduced in [**9**] where we have obtained growth theorems and estimated the distortion under the maps from H_c. In the the standard class S of all univalent functions $f(z) = z + a_2 z^2 + \dots$ in U the Koebe function $z(1 \pm z)^{-2}$ plays an important role. It turns out to be extremal in many problems, in particular, for estimates of the

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distortion in S. The function

$$k_b(z) = \frac{2bz}{1-z+\sqrt{(1-z)^2+4b^2z}} = bz + b(1-b^2)z^2 + \dots$$

plays the role of the Koebe function in the class H_c, where $b = 2c/(1+c^2)$ (see, e.g. [**9**]). It maps U onto the hyperbolic half-plane

$$U \setminus U\left(-\frac{1+c^2}{2c}, \frac{1-c^2}{2c}\right), \;\; c = \frac{1-\sqrt{1-b^2}}{b},$$

where $U(\zeta, r) = \{z \in \mathbb{C} : |z-\zeta| < r\}$. In [9] we got the sharp upper bound for $|f'(z)|$ for $|z| < k_b(\sqrt{2}-1)$ (note that $k_b(\sqrt{2}-1) \geqslant \sqrt{2}-1$). The extremal function is the canonical map $k_b(z)$ and its rotations. The lower estimate of $|f'(z)|$ has been achieved by an easier function $f(z) = cz$ for $|f(z)| \leqslant (\sqrt{2}-1)c$ (what is true at least for $|z| \leqslant (1+(2\sqrt{2}-1)c^2)/(3+2\sqrt{2}+c^2)$). We also derived in [**9**] a general two-point distortion theorem. The compact subclass $H(\alpha)$ of H with the inner normalization ($H(\alpha) = \{f \in H,\, f'(0) = \alpha\}$, $\alpha \in (0,1]$ is fixed) has been earlier introduced and considered by W. Ma, D. Minda, D. Mejía, Ch. Pommerenke, and the author in [**5**]–[**9**]. In [**9**] we also obtained the upper bound of $|f'(z)|$ in this class for $|z| < G(\alpha)$, where $G(\alpha)$ is a function of α with $G(\alpha) \geqslant \sqrt{2}-1$. Recently, W. Ma and D. Minda [**6**] repeated this result under a stronger condition $|z| \leqslant \sqrt{2}-1$ by a different method.

In this paper we are concerned with some distortion theorems making use of the extremal length method in the form of the moduli of families of curves and the reduced moduli. In particular, we obtain (with restriction $|z| \leqslant (1+3c^2)(3+c^2)$) the lower bound for $|f'(z)|$ in the class H_c.

2. Preliminaries

1. First, we observe that $f \in H$ (or H_c) if and only if the function $f^*(z) = e^{-i\theta}f(ze^{i\theta})$, $\theta \in [-\pi, \pi]$, is also from the same class. This transformation is known as *the rotation.*

A necessary and sufficient condition for an analytic function f to be h-convex functions obtained by W. Ma and D. Minda in [**5**] is of the form

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)}\frac{2|f(z)|^2}{1-|f(z)|^2}\right) > \frac{1-|z|}{1+|z|} > 0 \tag{1}$$

in U.

It easily follows from the results of W. Ma and D. Minda [**5**] that $c \leqslant |f'(0)| \leqslant b$ where the equality appears only for the functions $w = cz$ and $w = k_b^*(z)$, respectively, where k_b^* stands for rotations of k_b.

The functional $D_1(f,z) = \frac{f'(z)(1-|z|^2)}{1-|f(z)|^2}$ is of importance in the theory of h-convex functions. W. Ma and D. Minda [**5**] have given sharp estimates for $|D_1(f,z)|$ in the class $H(\alpha)$. The upper bound for $|D_1(f,z)|$ in the class H_c is similar as it has been shown in [**9**], but the lower bound is principally different, what we will observe further on. The same phenomenon seems to be true in other cases of standard functionals, such as $|f(z)|$, $|f'(z)|$, in the classes H_c and $H(\alpha)$. If an estimate is given by the canonical map in both classes the other one is also given by the canonical map for the class $H(\alpha)$

but by the map $w = cz$ for the class H_c. We should say that such simple extremal function does not imply similar simplicity of the proofs.

2. The method that we will use in the paper is based on the extremal length method in its form of the moduli method. We are concerned with a notion which appeared rather recently in [**2, 4, 13**] and, nowadays, is used for extremal problems for conformal maps (see [**12, 13, 14**]). It is called the *reduced modulus of a digon.* We define the reduced moduli of digons following E. G. Emel'yanov [**2**], G. V. Kuz'mina [**4**], and A. Yu. Solynin [**13**].

A *Stolz angle* at $\zeta \in \mathbb{C}$ is of the form

$$\Delta_\zeta(\psi, \theta, r) = \{z \in U(\zeta, r) : |\arg(z - \zeta) - \psi| < \theta\},$$

(see these definitions in [**11, 13**]).

Let D be a hyperbolic simply connected domain in $\mathbb{C}$ with two finite fixed boundary points a, b (maybe with the same support) on its boundary. All boundary elements, the neighbourhoods of which we consider, are assumed to be one-point. We call such D a *digon.* A hyperbolic simply connected domain in $\mathbb{C}$ with a single finite fixed boundary points a and a fixed arc l on the boundary, $a \notin l$, is called *a triangle.* The digon D has the *inner angle* φ_a with the vertex at a if $\varphi_a = 2 \sup \theta$, where the supremum is taken over all Stolz angles $\Delta_a(\psi, \theta, r)$ (the part which lies in D for r sufficiently small).

Let us set $S(a, \varepsilon)$, a connected component of $D \cap \{|z - a| < \varepsilon\}$, such that $a \in \partial S(a, \varepsilon)$. We denote by $D_{\varepsilon_1,\varepsilon_2}$ the domain $D \setminus \{S(a, \varepsilon_1) \cup S(b, \varepsilon_2)\}$ for sufficiently small ε_j, $j = 1, 2$. Let $M(D_{\varepsilon_1,\varepsilon_2})$ be the modulus of the family of arcs in $D_{\varepsilon_1,\varepsilon_2}$ joining the boundary arcs of $S(a, \varepsilon_1)$ and $S(b, \varepsilon_2)$ that lie in the circumferences $|z - a| = \varepsilon_1$ and $|z - b| = \varepsilon_2$. We choose the arcs so that each of them divides D into two triangles with their vertices at a and b and with the opposite legs on this arc. If the limit

$$m(D, a, b) = \lim_{\varepsilon_{1,2}\to 0} \left(\frac{1}{M(D_{\varepsilon_1,\varepsilon_2})} + \frac{1}{\varphi_a} \log \varepsilon_1 + \frac{1}{\varphi_b} \log \varepsilon_2 \right),$$

exists, then it is called the *reduced modulus of the digon* D, where $\varphi_a = \sup \Delta_a$, $\varphi_b = \sup \Delta_b$ are the inner angles, and the supremum is taken over all Stolz angles Δ_a and Δ_b inscribed in D at a or b respectively. The existence of the limit is a local characteristic ([**13**], Theorem 1.2) of the domain D. Suppose that there exists a conformal map $f(z)$ from the domain $S(a, \varepsilon_1) \subset D$ onto a circular sector, so that there exists the angular limit $f(a)$ which is the vertex of this sector and with the angle φ_a. If the function f has the angular finite non-zero derivative $f'(a)$ we say that the domain D is *conformal at the point* a. If the digon D is conformal at the points a, b, then the limit in the definition of $m(D, a, b)$ exists (see [**13**], Theorem 1.3).

Suppose that there exists a conformal map $f(z)$ of the digon D (which is conformal at a, b) onto a digon D', so that there exist the angular limits $f(a)$, $f(b)$ with the inner angles ψ_a and ψ_b at the vertices $f(a)$ and $f(b)$ which are also thought of as the supremum over all Stolz angles inscribed in D' with the vertices $w_1 = f(a)$ or $w_2 = f(b)$ respectively. If the function f has the angular finite non-zero derivatives $f'(a)$ and $f'(b)$, then $\varphi_a = \psi_{f(a)}$, $\varphi_b = \psi_{f(b)}$, and the reduced modulus exists and is changed by the rule

$$m(f(D), f(a), f(b)) = m(D, a, b) + \frac{1}{\psi_a} \log |f'(a)| + \frac{1}{\psi_b} \log |f'(b)|.$$

If we suppose, moreover, that f has the expansion

$$f(z) = w_1 + (z-a)^{\psi_a/\varphi_a}(c_1 + c_2(z-a) + \dots)$$

about the point a, and the expansion

$$f(z) = w_2 + (z-b)^{\psi_b/\varphi_b}(d_1 + d_2(z-a) + \dots)$$

about the point b, then the reduced modulus of D is changed by the rule

$$m(f(D), f(a), f(b)) = m(D, a, b) + \frac{1}{\psi_a} \log |c_1| + \frac{1}{\psi_b} \log |d_1|,$$

where c_1, d_1 are some complex non-zero constants. Obviously, one can extend this definition to the case of vertices with the infinite support.

3. We define now two problems about extremal partition of the disk $U_c = \{z : |z| < c\}$ by digons.

Let $0, B$ be punctures in U_c. We consider the family $\mathbb{D}_1$ of digons D in U_c such that $0, B \notin D$ are two vertices of any $D \in \mathbb{D}_1$. All digons are supposed to have the angles 2π at their vertices. We define the problem of minimizing the reduced modulus

$$\min_{D \in \mathbb{D}_1} m(D, 0, B)$$

There is a unique digon $D_1^* = U_c \setminus \{(-c, 0] \cup [|B|, c)\}$, rotated by the angle $\arg B$, that gives this minimum. The reduced modulus is calculated making use of a suitable conformal map of D_1^* onto the digon $\mathbb{C} \setminus [0, \infty)$ of modulus zero with respect to its vertices $0, \infty$.

$$m(D_1^*, 0, B) = \frac{1}{2\pi} \log \frac{c^2|B|^2}{c^2 - |B|^2}.$$

Now we consider the family $\mathbb{D}_2$ of digons D in U_c such that $0, B \notin D$ and B is a support of two vertices of any D with the inner angles π. We define the problem of minimizing the reduced modulus

$$\min_{D \in \mathbb{D}_2} m(D, B, B)$$

There is a unique digon $D_2^* = U_c \setminus [0, ce^{i \arg B})$ that gives this minimum. One can calculate this reduced modulus constructing a suitable conformal map as in the previous case

$$m(D_2^*, B, B) = \frac{2}{\pi} \log \frac{|B|(c + |B|)}{c - |B|}.$$

3. Lower estimations for distortion

THEOREM 3.1. *Let* $f \in H_c$, $|z| := r$.

(i) If

$$|z| \leqslant \frac{1+3c^2}{3+c^2},$$

then $\rho := |f(z)| \leqslant c$.

(ii) If $|f(z)| \leqslant c$, *then*

$$|f'(z)||f'(0)| \geqslant \left(\frac{\rho}{r}\right)^2 \frac{c^2(1-r^2)}{c^2-\rho^2} \geqslant c^2$$

with the obvious extremal function $f(z) \equiv cz$.

PROOF. Part (i) obviously follows from the inequality $|f(z)| \leqslant k_b(|z|)$. For (ii) we consider the digon $D^z = U \setminus \{(-1,0] \cup [r,1)\}$ with the vertices at 0 and r and with the reduced modulus

$$m(D^z, 0, r) = \frac{1}{2\pi} \log \frac{r^2}{1-r^2}.$$

Now we pose the problem of the extremal partition of U_c by the family of digons $\mathbb{D}_1$. Functions from H are starlike, therefore the function $|f(r)|$ increases in $r \in [0,1)$. Moreover, the disk $U_c \subset f(U)$. The domain $D = U_c \cap f(D^z)$ is a digon with the vertices at 0, $f(r)$ where it is conformal. It is admissible for the problem of minimizing the reduced modulus in the family $\mathbb{D}_1$ with $B = f(r)$.

Denote by $D^f = f(D^z)$, $D^f_\varepsilon = D^f \setminus \{(|w| < \varepsilon) \cup (|w - f(r)| < \varepsilon)\}$, $D_\varepsilon = D \setminus \{(|w| < \varepsilon) \cup (|w - f(r)| < \varepsilon)\}$ for a sufficiently small ε. Then

$$m(D^f, 0, f(r)) = \lim_{\varepsilon \to 0} \left(\frac{1}{M(D^f_\varepsilon)} + \frac{1}{2\pi} \log \varepsilon \right)$$

and

$$m(D, 0, f(r)) = \lim_{\varepsilon \to 0} \left(\frac{1}{M(D_\varepsilon)} + \frac{1}{2\pi} \log \varepsilon \right).$$

The quadrilaterals D^f_ε and D_ε have the common sides on the arcs of the circumferences $|w| = \varepsilon$ and $|w - f(r)| = \varepsilon$. Moreover, $D_\varepsilon \subset D^f_\varepsilon$. Therefore, $M(D^f_\varepsilon) \leqslant M(D_\varepsilon)$ and $m(D^f, 0, f(r)) \geqslant m(D, 0, f(r))$. This implies the following chain of inequalities:

$$m(D^z, 0, r) + \frac{1}{2\pi} \log |f'(r)||f'(0)| = m(f(D^z), 0, f(r)) \geqslant m(D, 0, f(r)) \geqslant \frac{1}{2\pi} \log \frac{c^2\rho^2}{c^2-\rho^2},$$

which is equivalent to the first inequality of (ii). The function $c^2\rho^2/(c^2-\rho^2)$ increases in ρ, thus, the inequality $\rho \geqslant cr$ finishes the whole proof. □

The consideration of the family of digons $\mathbb{D}_2$ and observations of the preceding proof lead to the following result.

THEOREM 3.2. *Let* $f \in H_c$, $|z| = r$. *If* $\rho := |f(z)| \leqslant c$, *then*

$$|f'(z)| \geqslant \left(\frac{\rho}{r}\right) \frac{(c+\rho)(1-r)}{(c-\rho)(1+r)} \geqslant c, \qquad |D_1(f,z)| \geqslant \frac{c(1-r^2)}{1-c^2r^2},$$

with the obvious extremal function $f(z) \equiv cz$.

For $|f(z)| > c$, the inequality $\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2}$ (see [7]) implies the obvious estimates

$$|f'(z)| > \frac{c}{2r} \quad \text{and} \quad |D_1(f,z)| > \frac{(1-r^2)c}{2r(1-c^2r^2)}$$

which are not sharp, thus, the problem in the rest of the unit disk is still open.

4. Two-point distortion

A two-variable characterization of functions from H have been obtained by Ch. Pommerenke, D. Mejía [7] and W. Ma, D. Minda [6]. In this section we collect some results about two-point distortion under h-convex maps. The first theorem is obtained by the author in [9].

THEOREM 4.1. [9] *Let $f \in H_c$ and $0 < r_1 < r_2 < 1$. Then*
(i)

$$\frac{\rho_2-\rho_1}{1-\rho_2\rho_1} \leqslant \frac{r_2-r_1}{1-r_2r_1}\,\frac{b+\frac{\rho_2+\rho_1}{1+\rho_2\rho_1}}{1+b\frac{\rho_2+\rho_1}{1+\rho_2\rho_1}},$$

$$\textit{where} \quad \rho_1 = |f(r_1)|, \rho_2 = |f(r_2)|, \quad b = \frac{2c}{1+c^2}.$$

(ii) If $-1 < -r_1 \leqslant 0 < r_2 < 1$, and $\rho_1, \rho_2 < c$ (this is true at least for $r_1, r_2 < \frac{1+3c^2}{3+c^2}$), then

$$\frac{\rho_2+\rho_1}{c^2+\rho_2\rho_1} \geqslant \frac{r_2+r_1}{c(1+r_2r_1)}.$$

In part (ii) we write $\rho_1 = |f(-r_1)|$. The equality appears for the functions $w = k_b^*(z)$ in (i) and $w = cz$ in (ii) respectively.

From this theorem, in particular, it follows that for $f \in H_c$ the sharp lower estimate of the growth is $c|z| \leqslant |f(z)|$. This result shows the qualitative difference of the class H_c from $H(\alpha)$, where the lower estimation is given by the canonical map. The sharp upper bound in H_c is given as follows $|f(z)| \leqslant k_b(|z|)$. The equality appears for the functions $w = cz$ and $w = k_b^*(z)$ respectively. If $d(\cdot,\cdot)$ is the Poincaré metric in U, then from i) it follows that

$$d(|f(r_1)|, |f(r_2)|) \leqslant d(k_b(r_1), k_b(r_2)).$$

Another result connected with Theorem 4.1 states (see [**9**]) that

$$|D_1(f,z)| \leqslant \frac{b\rho^2+2\rho+b}{\rho^2+2b\rho+1},$$

where $\rho = |f(z)|$. The inequality is sharp for the function $k_b^*(z)$.

In the class of all holomorphic univalent in U functions with normalization $f(0) = 0$ we have the inequality (see [10])

$$\frac{\rho_2}{\rho_1} \leqslant \frac{r_2(1-r_1)^2}{r_1(1-r_2)^2},$$

for $\rho_1 = |f(r_1)|$, $\rho_2 = |f(r_2)|$ and $0 < r_1 < r_2 < 1$.

The restriction $|f(z)| < 1$ in U leads to the following inequality

$$\frac{\rho_2}{\rho_1} \leqslant \left(\frac{(1-\rho_2)(1-r_1)}{(1-\rho_1)(1-r_2)}\right)^2 \frac{r_2}{r_1}.$$

The inequality $cr \leqslant |f(r)| \leqslant r$ immediately implies a rough estimate of the ratio $|f(r_2)|/|f(r_1)| \leqslant r_2/c\,r_1$, $0 < r_1 < r_2 < 1$. For the sharp form of this we have the following theorem.

THEOREM 4.2. *Let $f \in H_c$ and $0 < r_1 < r_2 < 1$. Then (i)*

$$\sqrt{\frac{r_2}{r_1}} < \frac{\rho_2}{\rho_1} \leqslant \frac{(b\rho_1+1)(\rho_1+b)(1+\rho_2)^2}{(b\rho_2+1)(\rho_2+b)(1+\rho_1)^2}\left(\frac{(1-\rho_2)(1-r_1)}{(1-\rho_1)(1-r_2)}\right)^2 \frac{r_2}{r_1}.$$

for $\rho_1 = |f(r_1)|, \rho_2 = |f(r_2)|$. The right-hand side is sharp and the equality appears for the function $w = k_b^*(z)$.

(ii) If $-1 < -r_1 \leqslant 0 < r_2 < 1$, $\rho_1, \rho_2 < c$ (this is true at least for $r_1, r_2 < \frac{1+3c^2}{3+c^2}$), then

$$\frac{\rho_1(1-\rho_2^2)^2(1-b\rho_1)(b-\rho_1)}{(\rho_2+\rho_1)(1+\rho_1\rho_2)(1+b(\rho_2-\rho_1)-\rho_1\rho_2)^2} \geqslant \frac{r_1(1-r_2)^2}{(r_2+r_1)(1+r_1r_2)}.$$

In the part (ii) we write $\rho_1 = |f(-r_1)|$. The equality appears for the function $w = k_b^*(z)$.

PROOF. Let $U' = U \setminus \{0, r_1, r_2\}$ and Γ be a family of Jordan closed curves that separate in U' the points r_1, r_2 from 0 and ∂U which are homotopic on U' to the slit along the segment $[r_1, r_2]$. We denote by $m(\Gamma)$ the modulus of this family. Let G be a doubly connected hyperbolic domain in U' associated with Γ, i.e. each closed curve separating the boundary components of G is from Γ. Let us denote by $M(G)$ the modulus of G with respect to the family of curves in G separating its boundary components. Then, it is known [**3**] that $M(G) \leqslant M(D) = m(\Gamma)$, where $D = U \setminus \{(-1, 0] \cup [r_1, r_2]\}$. Moreover, $m(\Gamma) = \frac{1}{2}\frac{\mathbf{K}}{\mathbf{K}'}\left(\sqrt{\frac{r_1}{r_2}\frac{1-r_2}{1-r_1}}\right)$, where $\mathbf{K}(k)$ and $\mathbf{K}'(k) = \mathbf{K}(\sqrt{1-k^2})$ are the conjugated complete elliptic integrals.

Since functions from H_c are starlike, the inequality

$$|f(r_1)| < |f(r_2)| \tag{2}$$

holds for $r_1 < r_2$.

We denote by D^* the result of circular symmetrization of the domain $f(D)$ with respect to the positive real axis $\mathbb{R}^+$. Since $f \in H_c$, the domain D^* lies in U within the domain $k_b(U)$. By the inequality (2) we have $\rho_1 < \rho_2$ and $D^* \subset \widetilde{D}(\rho_1, \rho_2) := U \setminus (U_c \cup [-c, 0] \cup [\rho_1, \rho_2])$. Hence, $M(D) = M(f(D)) \leqslant M(D^*) \leqslant M(\widetilde{D})$. By a suitable conformal maps we calculate

$$M(\widetilde{D}(\rho_1, \rho_2)) = \frac{1}{2}\frac{\mathbf{K}}{\mathbf{K}'}\left(\sqrt{\frac{\rho_1(b\rho_1+1)(\rho_1+b)}{\rho_2(b\rho_2+1)(\rho_2+b)}\frac{1-\rho_2^2}{1-\rho_1^2}}\right).$$

The function $\frac{\mathbf{K}}{\mathbf{K}'}(k)$ increases in $k \in (0,1)$. This leads to the inequality in the right-hand side in (i). The function $k_b(z)$ obviously maps D onto $\widetilde{D}(k_b(r_1), k_b(r_2))$ and gives the equality in the above chain of inequalities. This yields the statement about the extremality of $k_b(z)$ and finishes the proof of the right-hand side inequality of (i).

Due to the result by D. Mejía, Ch. Pommerenke [7] hyperbolically convex univalent functions are starlike of the order 1/2. Hence,

$$\frac{d|f(r)|}{dr} = |f(r)|\mathrm{Re}\ \frac{f'(r)}{f(r)} \geqslant \frac{|f(r)|}{2r}. \tag{3}$$

Integrating the inequality (3) we deduce the left-hand side inequality in (i) of Theorem 4.2.

Now we consider the family of Jordan closed curves on $U'' = U \setminus \{0, -r_1, r_2\}$ that separate in U'' the points $0, r_2$ from $-r_1$ and ∂U. The corresponding modulus of Γ is equal to

$$m(\Gamma) = \frac{1}{2}\frac{\mathbf{K}}{\mathbf{K}'}\left(\sqrt{\frac{r_1(1-r_2)^2}{(r_2+r_1)(1+r_1r_2)}}\right).$$

Further similar application of symmetrization leads to the inequality in (ii).

□

In [**12**] we obtained the following result. If f is a map from U into U, $f(0) = 0$, and the angular limit $f(1) = 1$ exists as well as the angular non-zero, finite derivative $f'(1)$, then $|f'(1)| \geqslant 1/\sqrt{|f'(0)|} > 1$. For the h-convex functions we have the following result.

COROLLARY 4.1. *Let $f \in H_c$ and the angular limit $f(1) = 1$ exists as well as the angular derivative $f'(1)$. Then*

$$|f'(1)| \geqslant \frac{1+b}{2\sqrt{b|f'(0)|}} \geqslant \frac{1+b}{2b} = \frac{(1+c)^2}{4c}.$$

The equality signs appear for the function $w = k_b^*(z)$.

This partially means that in the whole class H all functions with $f(1) = 1$ satisfy the inequality $|f'(1)| \geqslant 1$.

COROLLARY 4.2. [9] *Let $f \in H_c$. Then*

$$\left|\frac{zf'(z)}{f(z)}\right| \leqslant \frac{1+|z|}{1-|z|}\frac{(1+b\rho)(\rho+b)(1-\rho^2)}{(\rho^2+2b\rho+1)(b\rho^2+2\rho+b)},$$

where $\rho = |f(z)|$. The inequality is sharp for the function $k_b^*(z)$.

The inequality in the last corollary has been obtained in ([**9**], Theorem 4) and led to the sharp form of the distortion theorem. For $|z| \leqslant k_b^{-1}(\sqrt{2}-1) =: F(c)$ we have $|f'(z)| \leqslant k_b'(|z|)$ with the equality sign for the function $k_b^*(z)$.

THEOREM 4.3. *Let $f \in H_c$. Then for $0 < r_1 < r_2 < 1$ and $|z| = 1$*

$$\left|\frac{D_1(f, r_2z)}{D_1(f, r_1z)}\right| \geqslant \frac{(1-r_2)(1+r_1)}{(1+r_2)(1-r_1)}.$$

PROOF. We calculate

$$\frac{\partial}{\partial r}\frac{|f'(rz)|}{1-|f(rz)|^2} = \frac{1}{r}\frac{|f'(rz)|}{1-|f(rz)|^2}\mathrm{Re}\ \left(\frac{rzf''(rz)}{f'(rz)} + 2\frac{rzf'(rz)}{f(rz)}\frac{|f(rz)|^2}{1-|f(rz)|^2}\right) \geqslant$$

$$\geqslant -\frac{2}{1+r}\frac{|f'(rz)|}{1-|f(rz)|^2}.$$

The last inequality has appeared due to the characterization for hyperbolically convex functions (1) given in [5]. Integrating this inequality from r_1 up to r_2 we obtain the inequality in the theorem. □

By analogy with [**7**] one can also deduce some simple consequences from the estimations of the expression Re $\frac{zf'(z)}{f(z)}$ for the functions from H_c, $0 < r_1 < r_2 < 1$,

$$|f(r_2) - f(r_1)| > \frac{c}{2}(r_2 - r_1).$$

If, moreover, $r_2 < k_b^{-1}(\sqrt{2}-1)$, then

$$|f(r_2) - f(r_1)| < \frac{(1+c)^2}{4c}(r_2 - r_1).$$

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