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The integrals in Gradshteyn and Ryzhik. Part 33: Sines and cosines of multiple and of linear and more complicated functions of the argument

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Abstract. The table of Gradshteyn and Ryzhik contains many integrals that involve trigonometric functions. Some examples are discussed.

1. Introduction

The table of integrals by I. S. Gradshteyn and I. M. Ryzhik [**5**] contains a large selection of integrals containing trigonometric functions. The evaluation of some of them has appeared earlier papers in this collection, such as [**1, 2, 3, 4, 9**]. Naturally, evaluation of integrals of this type appear throughout the literature. As examples we cite [**6, 7, 10, 11**].

The goal of this work is to present all proofs of the entries in Sections 2.53 and 2.54 in [**5**]. This is a continuation of the program, initiated with [**8**] with the purpose of providing proofs to all entries in the cited table. The definitions of $\sin x$ and $\cos x$ leads directly to

$$\int \sin x \, dx = -\cos x \quad \text{and} \quad \int \cos x \, dx = \sin x. \tag{1.1}$$

Only the elementary properties of the trigonometric functions, such as the addition formulas

$$\sin(u+v) = \sin u \cos v + \cos u \sin v \quad \cos(u+v) = \cos u \cos v - \sin u \sin v, \tag{1.2}$$

and their consequences are used. All the proofs are given in complete detail.

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2. Entries in Section 2.531

This section contains two elementary entries.

2.1. Entry 2.531.1.

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) \tag{2.1}$$

PROOF. The change of variables $t = ax+b$ gives

$$\int \sin(ax+b) = \frac{1}{a}\int \sin t\,dt \tag{2.2}$$

and the result follows from (1.1). □

2.2. Entry 2.531.2.

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) \tag{2.3}$$

PROOF. As in the proof of (2.1), let $t = ax+b$ and use (1.1) to obtain the result. □

3. Entries in Section 2.532

3.1. Entry 2.532.1.

$$\int \sin(ax+b)\sin(cx+d)\,dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} - \frac{\sin[(a+c)x+b+d]}{2(a+c)} \tag{3.1}$$

provided $a^2 \neq c^2$.

PROOF. The identity $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$ gives

$$\int \sin(ax+b)\sin(cx+d)\,dx = \frac{1}{2}\int \left[\cos[(a-c)x+(b-d)] - \cos[(a+c)x+(b+d)]\right]\,dx.$$

The result now follows from Entry 2.531.2. □

3.2. Entry 2.532.2.

$$\int \sin(ax+b)\cos(cx+d)\,dx = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)} \tag{3.2}$$

provided $a^2 \neq c^2$.

PROOF. The proof follows as in Entry 2.532.1 using the identity

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]. \tag{3.3}$$

□

3.3. Entry 2.532.3.

$$\int \cos(ax+b)\cos(cx+d)\,dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} + \frac{\sin[(a+c)x+b+d]}{2(a+c)} \tag{3.4}$$

provided $a^2 \neq c^2$.

Proof. As before, now using $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$. □

The cases with $c = a$ are given below:

3.4. Entry 2.532.4.

$$\int \sin(ax+b)\sin(ax+d)\,dx = \frac{x}{2}\cos(b-d) - \frac{\sin[2ax+b+d]}{4a}. \tag{3.5}$$

Proof. The identity $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$ yields

$$\sin(ax+b)\sin(ax+d) = \tfrac{1}{2}\left[\cos(b-d)-\cos(2ax+b+d)\right] \tag{3.6}$$

and now integrate using (1.1). □

3.5. Entry 2.532.5.

$$\int \sin(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\sin(b-d) - \frac{\cos(2ax+b+d]}{4a}. \tag{3.7}$$

Proof. The proof follows from integrating

$$\sin(ax+b)\cos(ax+d) = \tfrac{1}{2}\left[\sin(b-d)+\sin(2ax+b+d)\right].$$

□

3.6. Entry 2.532.6.

$$\int \cos(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\cos(b-d) + \frac{\sin(2ax+b+d]}{4a}. \tag{3.8}$$

Proof. Integrate the identity

$$\cos(ax+b)\cos(ax+d) = \tfrac{1}{2}\,]\cos(b-d)-\cos(2ax+b+d)].$$

□

4. Entries in Section 2.533

4.1. Entry 2.533.1.

$$\int \sin ax\cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \tag{4.1}$$

provided $a^2 \neq b^2$.

Proof. Integrate

$$\sin ax\cos bx = \tfrac{1}{2}\left[\sin((a+b)x)+\sin((a-b)x)\right].$$

□

4.2. Entry 2.533.2.

$$\int \sin ax \sin bx \sin cx \, dx = -\frac{1}{4}\left\{ \frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \right\} \tag{4.2}$$

PROOF. Start with the trigonometric expression

$$T := \sin A \sin B \sin C = (\sin A \sin B) \sin C \tag{4.3}$$

to obtain

$$\begin{aligned} T &= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)\sin C \\ &= \frac{1}{2}\left(\cos(A-B)\sin C - \cos(A+B)\sin C\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\left(\sin(A-B+C) + \sin(-A+B+C)\right) - \frac{1}{2}\left(\sin(A+B+C) + \sin(-A-B+C)\right)\right) \\ &= \frac{1}{4}\left(\sin(A-B+C) + \sin(-A+B+C) - \sin(A+B+C) - \sin(-A-B+C)\right). \end{aligned}$$

Now put $A = ax$, $B = bx$, $C = cx$ and integrate to obtain the result. □

4.3. Entry 2.533.3.

$$\int \sin ax \cos bx \cos cx \, dx = -\frac{1}{4}\left\{ \frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \right\} \tag{4.4}$$

PROOF. Proceed as in Entry **2.533.2** using the identity

$$4 \sin A \cos B \cos C = \sin(A+B+C) - \sin(B+C-A) + \sin(A+B-C) + \sin(A+C-B).$$

□

4.4. Entry 2.533.4.

$$\int \cos ax \sin bx \sin cx \, dx = \frac{1}{4}\left\{ \frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \right\} \tag{4.5}$$

PROOF. Proceed as in Entry **2.533.2** now using the identity

$$4 \cos A \sin B \sin C = \cos(A+B-C) + \cos(A+C-B) - \cos(A+B+C) - \cos(B+C-A).$$

□

4.5. Entry 2.533.5.

$$\int \cos ax \cos bx \cos cx \, dx = \frac{1}{4}\left\{ \frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \right\} \tag{4.6}$$

PROOF. Finally, proceed as in Entry **2.533.2** now using the identity

$$4\cos A \cos B \cos C = \cos(A+B+C) + \cos(B+C-A) + \cos(A-B+C) - \cos(A+B-C).$$

□

5. Entries in Section 2.534

5.1. Entry 2.534.1.

$$\int \frac{\cos px + i \sin px}{\sin nx} \, dx = -2 \int \frac{z^{p+n-1}}{1-z^{2n}} \, dz \tag{5.1}$$

where $z = \cos x + i \sin x$.

PROOF. Write $z = e^{ix}$ to obtain

$$\begin{aligned} \int \frac{z^{p+n-1}}{1-z^{2n}} \, dz &= i \int \frac{e^{i(p+n-1)x}}{1-e^{2inx}} e^{ix} \, dx \\ &= -i \int \frac{e^{ipx}}{e^{inx}-e^{-inx}} \, dx \\ &= -\frac{1}{2} \int \frac{e^{ipx}}{\sin nx} \, dx, \end{aligned} \tag{5.2}$$

as claimed. □

5.2. Entry 2.534.2.

$$\int \frac{\cos px + i \sin px}{\cos nx} \, dx = -2i \int \frac{z^{p+n-1}}{1+z^{2n}} \, dz \tag{5.3}$$

where $z = \cos x + i \sin x$.

PROOF. Write $z = e^{ix}$ to obtain, as in proof of entry **2.534.1**

$$\int \frac{z^{p+n-1}}{1+z^{2n}} \, dz = i \int \frac{e^{i(p+n)x} \, dx}{1+e^{2nix}}. \tag{5.4}$$

A simple manipulation now gives the result. This entry has a typo: the denominator appears as $1 - z^{2n}$. □

6. Entries in Section 2.539

6.1. Entry 2.539.1.

$$\int \frac{\sin(2n+1)x}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\sin 2kx}{2k} + x \tag{6.1}$$

Proof. Differentiation shows that the problem is equivalent to the identity

$$\frac{\sin(2n+1)x}{\sin x} = 2\sum_{k=1}^{n}\cos(2kx) + 1. \tag{6.2}$$

To prove this, start with

$$\begin{aligned} \sum_{k=1}^{n}\cos(kx) &= \text{Re}\left(\sum_{k=1}^{n} e^{ikx}\right) \\ &= \text{Re}\left(\frac{e^{i(n+1)x}-1}{e^{ix}-1} - 1\right) \\ &= \text{Re}\left(\frac{e^{i(n+1)x}-1}{e^{ix/2}(e^{ix/2}-e^{-ix/2})} - 1\right) \\ &= \text{Re}\left(\frac{e^{i(n+1/2)x}-e^{-ix/2}}{e^{ix/2}-e^{-ix/2}} - 1\right) \\ &= \frac{\sin(n+\frac{1}{2})x + \sin(\frac{x}{2})}{2\sin(\frac{x}{2})} - 1 \\ &= \frac{\sin(n+\frac{1}{2})x - \sin\frac{x}{2}}{2\sin(\frac{x}{2})}. \end{aligned} \tag{6.3}$$

Now replace x by $2x$ and use the identity

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \tag{6.4}$$

to obtain

$$\sum_{k=1}^{n}\cos(2kx) = \frac{\sin(nx)\cos((n+1)x)}{\sin x} = \frac{1}{2}\left(\frac{\sin(2n+1)x}{\sin x} - 1\right) \tag{6.5}$$

using $\sin(nx)\cos((n+1)x) = \frac{1}{2}\left[\sin(2n+1)x - \sin x\right]$. Integration leads to the stated formula. □

6.2. Entry 2.539.2.

$$\int \frac{\sin 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n}\frac{\sin(2k-1)x}{2k-1} \tag{6.6}$$

Proof. Differentiating the right-hand side shows that one needs to evaluate

$$\sum_{k=1}^{n}\cos(2k-1)x = \text{Re}\left(\sum_{k=1}^{n} e^{i(2k-1)x}\right). \tag{6.7}$$

Using the value of the geometric sum and some simplification gives

$$\sum_{k=1}^{n} e^{i(2k-1)x} = \frac{e^{2ix} - e^{2i(n+1)x}}{e^{ix}(1-e^{2ix})} = \frac{1-e^{2inx}}{-2i\sin x}.$$

Taking the real part shows that

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin(2nx)}{2\sin x}. \tag{6.8}$$

Now integrate to get the result. □

6.3. Entry 2.539.3.

$$\int \frac{\cos(2n+1)x}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\cos 2kx}{2k} + \ln\sin x \tag{6.9}$$

Proof. The evaluation requires the computation of

$$\begin{aligned} \sum_{k=1}^{n} \sin(2kx) &= \text{Im}\left(\sum_{k=1}^{n} e^{2ikx}\right) \\ &= \text{Im}\left(\frac{1-e^{2(n+1)ix}}{1-e^{2ix}}\right) \\ &= \text{Im}\left(\frac{e^{(2n+1)ix}-e^{-ix}}{e^{ix}-e^{-ix}}\right) \\ &= \frac{-\cos(2n+1)x+\cos x}{2\sin x}. \end{aligned} \tag{6.10}$$

The result follows by integration. □

6.4. Entry 2.539.4.

$$\int \frac{\cos 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n} \frac{\cos(2k-1)x}{2k-1} + \ln\tan\frac{x}{2} \tag{6.11}$$

Proof. This entry follows directly by integration of the identity

$$\sum_{k=1}^{n} \sin(2k-1)x = \frac{1-\cos(2nx)}{2\sin x}. \tag{6.12}$$

This is establish using the method described in the proof of formula (6.11). □

6.5. Entry 2.539.5.

$$\int \frac{\sin(2n+1)x}{\cos x}\,dx = 2\sum_{k=1}^{n} (-1)^{n-k+1}\frac{\cos 2kx}{2k} + (-1)^{n+1}\ln\cos x \tag{6.13}$$

Proof. The proof follows directly by integrating the expression

$$\sum_{k=1}^{n} (-1)^k \sin(2kx) = \frac{(-1)^n\sin(2n+1)x - \sin x}{2\cos x}. \tag{6.14}$$

The proof of this identity follows the usual procedure. □

6.6. Entry 2.539.6.

$$\int \frac{\sin 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n}(-1)^{n-k+1}\frac{\cos(2k-1)x}{2k-1} \tag{6.15}$$

PROOF. The proof follows directly from the formula

$$\sum_{k=1}^{n}(-1)^k \sin(2k-1)x = \frac{(-1)^n \sin(2nx)}{2\sin x}, \tag{6.16}$$

establish by the procedure indicated in the proof of (6.9). □

6.7. Entry 2.539.7.

$$\int \frac{\cos(2n+1)x}{\cos x}\,dx = 2\sum_{k=1}^{n}(-1)^{n-k}\frac{\sin 2kx}{2k} + (-1)^n x \tag{6.17}$$

PROOF. The proof is a direct consequence of the identity

$$\sum_{k=1}^{n}(-1)^k \cos(2kx) = \frac{(-1)^n \cos(2n+1)x}{2\cos x} - \frac{1}{2}, \tag{6.18}$$

established by the familiar procedure. □

6.8. Entry 2.539.8.

$$\int \frac{\cos 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n}(-1)^{n-k}\frac{\sin(2k-1)x}{2k-1} + (-1)^n \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \tag{6.19}$$

PROOF. The proof follows from the identity

$$\sum_{k=1}^{n}(-1)^k \cos(2k-1)x = \frac{(-1)^n \cos 2nx - 1}{2\cos x}, \tag{6.20}$$

established as before. □

7. Entries in Section 2.541

7.1. Entry 2.541.1.

$$\int \sin(n+1)x \sin^{n-1} x\,dx = \frac{1}{n}\sin^n x\,\sin nx \tag{7.1}$$

PROOF. The entry can be written as

$$\int \sin(n+1)x\sin^{n-1}x\,dx = \int \sin(nx)\cos x\,(\sin x)^{n-1}\,dx + \int \cos x(\sin x)^n\,dx. \tag{7.2}$$

Integrate the first term by parts, with $u = \sin(nx)$ and $dv = \cos x(\sin x)^{n-1}\,dx$ to obtain

$$\int \sin(nx)\frac{d}{dx}\frac{(\sin x)^n}{n}\,dx = \sin(nx)\frac{(\sin x)^n}{n} - \int \cos x(\sin x)^n\,dx. \tag{7.3}$$

Replacing in (7.2) gives the result. ☐

7.2. Entry 2.541.2.

$$\int \sin(n+1)x \cos^{n-1} x\, dx = -\frac{1}{n} \cos^n x \, \cos nx \tag{7.4}$$

Proof. Expand $\sin(n+1)x = \sin nx \cos x + \cos nx \sin x$ to obtain

$$\int \sin(n+1)x \cos^{n-1} x\, dx = \int \sin nx \cos^n x\, dx + \int \cos nx \sin x \cos^{n-1} x\, dx. \tag{7.5}$$

Integrate by parts the first integral with $u = \cos^n x$ and $dv = \sin nx\, dx$, to produce

$$\int \sin nx \cos^n x\, dx = -\frac{1}{n} \cos^n x \cos nx - \int \cos nx \cos^{n-1} x \sin x\, dx. \tag{7.6}$$

Then (7.5) gives the result. ☐

7.3. Entry 2.541.3.

$$\int \cos(n+1)x \sin^{n-1} x\, dx = \frac{1}{n} \sin^n x \, \cos nx \tag{7.7}$$

Proof. The addition formula for $\cos x$ gives

$$\int \cos(n+1)x \sin^{n-1} x\, dx = \int \cos nx \cos x \sin^{n-1} x\, dx - \int \sin nx \sin^n x\, dx. \tag{7.8}$$

Integrate by parts the first integral on the right, with $u = \cos nx$ and $dv = \cos x \sin^{n-1} x$ to obtain the result, making use of the cancellation of two integrals. ☐

7.4. Entry 2.541.4.

$$\int \cos(n+1)x \cos^{n-1} x\, dx = \frac{1}{n} \cos^n x \, \sin nx \tag{7.9}$$

Proof. The addition formula for $\cos x$ gives

$$\int \cos(n+1)x \sin^{n-1} x\, dx = \int \cos nx \cos x \sin^{n-1} x\, dx - \int \sin nx \sin^n x\, dx. \tag{7.10}$$

Integrate by parts the first integral on the right, with $u = \cos^n x$ and $dv = \cos nx$ to obtain the result, making use of the cancellation of two integrals. ☐

7.5. Entry 2.541.5.

$$\int \sin[(n+1)(\tfrac{\pi}{2} - x)] \sin^{n-1} x\, dx = \frac{1}{n} \sin^n x \, \cos n \left(\tfrac{\pi}{2} - x \right) \tag{7.11}$$

Proof. The expansion of $\sin[(n+1)(\frac{\pi}{2} - x)]$ gives

$$\int \sin[(n+1)(\tfrac{\pi}{2} - x)] \sin^{n-1} x\, dx = \int \sin\left[n\left(\tfrac{\pi}{2} - x\right)\right] \cos x \sin^{n-1} x\, dx + \int \cos\left[n\left(\tfrac{\pi}{2} - x\right)\right] \sin^n x\, dx. \tag{7.12}$$

Integrate the first new integral by parts with $u = \sin\left[n\left(\frac{\pi}{2} - x\right)\right]$ and $dv = \sin^{n-1} x \, \cos x \, dx$. This yields the second new integral, with the opposite sign. This cancellation completes the proof. □

7.6. Entry 2.541.6.

$$\int \cos[(n+1)(\tfrac{\pi}{2} - x)] \sin^{n-1} x \, dx = -\frac{1}{n} \sin^n x \, \sin n\left(\tfrac{\pi}{2} - x\right) \tag{7.13}$$

PROOF. Proceed as in the previous entry using the expansion

$$\cos[(n+1)(\tfrac{\pi}{2} - x)] \sin^{n-1} x = \cos\left[n\left(\tfrac{\pi}{2} - x\right)\right] \cos x \sin^{n-1} x - \sin\left[n\left(\tfrac{\pi}{2} - x\right)\right] \sin^n x. \tag{7.14}$$

□

8. Entries in Section 2.542

8.1. Entry 2.542.1.

$$\int \frac{\sin 2x}{\sin^n x} \, dx = -\frac{2}{(n-2)\sin^{n-2} x} \tag{8.1}$$

for $n \neq 2$.

PROOF. The double angle formula $\sin(2x) = 2 \sin x \cos x$ yields

$$\int \frac{\sin 2x}{\sin^n x} \, dx = 2 \int \sin^{1-n} x \, \cos x \, dx. \tag{8.2}$$

The change of variables $u = \sin x$ now produces

$$2 \int \sin^{1-n} x \, \cos x \, dx = 2 \int u^{1-n} \, du = \frac{2}{2-n} u^{2-n}, \tag{8.3}$$

completing the proof. □

8.2. Entry 2.542.2.

$$\int \frac{\sin 2x}{\sin^2 x} \, dx = 2 \ln \sin x \tag{8.4}$$

PROOF. The relation $\sin(2x) = 2 \sin x \cos x$ and the change of variables $u = \sin x$ give

$$\int \frac{\sin 2x}{\sin^2 x} \, dx = 2 \int \frac{\cos x}{\sin x} \, dx = 2 \int \frac{du}{u} = 2 \ln u. \tag{8.5}$$

This confirms the result. □

9. Entries in Section 2.543

9.1. Entry 2.543.1.

$$\int \frac{\sin 2x}{\cos^n x}\, dx = \frac{2}{(n-2)\cos^{n-2} x} \tag{9.1}$$

for $n \neq 2$.

PROOF. The fact that $\frac{d}{dx}\cos^2 x = -2\cos x \sin x = -\sin(2x)$ suggests the change of variables $u = \cos^2 x$. Then

$$\int \frac{\sin 2x}{\cos^n x}\, dx = -\int u^{-n/2}\, du = -\frac{1}{1-n/2} u^{1-n/2}. \tag{9.2}$$

This simplifies to the expression given above. □

9.2. Entry 2.543.2.

$$\int \frac{\sin 2x}{\cos^2 x}\, dx = -2 \ln \cos x \tag{9.3}$$

PROOF. As in the proof of entry **2.543.1** let $u = \cos^2 x$ to obtain

$$\int \frac{\sin 2x}{\cos^2 x}\, dx = -\int \frac{du}{u}, \tag{9.4}$$

and this completes this evaluation. □

10. Entries in Section 2.544

10.1. Entry 2.544.1.

$$\int \frac{\cos 2x}{\sin x} dx = 2\cos x + \ln \tan \frac{x}{2} \tag{10.1}$$

PROOF. Start with $\cos 2x = 1 - 2\sin^2 x$ so that

$$\int \frac{\cos 2x}{\sin x} dx = \int \frac{dx}{\sin x} - 2 \int \sin x \, dx. \tag{10.2}$$

The first integral is evaluated using Weierstrass substitution $t = \tan(x/2)$. The expressions

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad dx = \frac{2dt}{1+t^2} \tag{10.3}$$

then give

$$\int \frac{dx}{\sin x} = \int \frac{dt}{t} = \ln t = \ln \tan \frac{x}{2}. \tag{10.4}$$

This completes the evaluation. □

10.2. Entry 2.544.2.

$$\int \frac{\cos 2x}{\sin^2 x} dx = -\cot x - 2x \tag{10.5}$$

Proof. Start with

$$\frac{\cos 2x}{\sin^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x} = \cot^2 x - 1 = \text{cosec}^2 x - 2. \tag{10.6}$$

Now use the elementary formula

$$\frac{d}{dx}\cot\, x = -\text{cosec}^2 x \tag{10.7}$$

to complete the evaluation. □

10.3. Entry 2.544.3.

$$\int \frac{\cos 2x}{\sin^3 x} dx = -\frac{\cos x}{2\sin^2 x} - \frac{3}{2}\ln\tan\frac{x}{2} \tag{10.8}$$

Proof. The Weierstrass change of variables (10.3) gives, with $t = \tan(x/2)$,

$$\begin{aligned} \int \frac{\cos 2x}{\sin^3 x} dx &= \int \frac{t^4 - 6t^2 + 1}{4t^3}\, dt \\ &= \frac{1}{8}t^2 - \frac{3}{2}\ln t - \frac{1}{8t^2}. \end{aligned} \tag{10.9}$$

Now use the identity

$$\tan^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{1+\cos x} \tag{10.10}$$

to complete the evaluation. □

10.4. Entry 2.544.4.

$$\int \frac{\cos 2x}{\cos x} dx = 2\sin x - \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \tag{10.11}$$

Proof. The identity $\cos 2x = 2\cos^2 x - 1$ leads to

$$\int \frac{\cos 2x}{\cos x}\, dx = 2\int \cos x\, dx - \int \frac{dx}{\cos x} \tag{10.12}$$

and each of the integral can be evaluated by elementary methods to confirm the result. □

10.5. Entry 2.544.5.

$$\int \frac{\cos 2x}{\cos^2 x} dx = 2x - \tan x \tag{10.13}$$

Proof. The identity

$$\frac{\cos 2x}{\cos^2 x} = 1 - \tan^2 x = 2 - \sec^2 x \tag{10.14}$$

and the result follows from $\frac{d}{dx}\tan x = \sec^2 x$. □

10.6. Entry 2.544.6.

$$\int \frac{\cos 2x}{\cos^3 x} dx = -\frac{\sin x}{2\cos^2 x} + \frac{3}{2} \ln \tan\left(\frac{\pi}{4} + \frac{x}{2} \right) \tag{10.15}$$

PROOF. The identity $\cos 2x = 1 - 2\sin^2 x$ leads to

$$\int \frac{\cos 2x}{\cos^3 x}\, dx = 2 \int \frac{dx}{\cos x} - \int \frac{dx}{\cos^3 x}. \tag{10.16}$$

The first integral is evaluated using the Weiestrass substitution $t = \tan x/2$ to obtain

$$\int \frac{dx}{\cos x} = \int \frac{2\, dt}{1-t^2}, \tag{10.17}$$

and this is evaluated using the partial fraction expansion

$$\frac{2}{1-t^2} = \frac{1}{t+1} - \frac{1}{t-1}. \tag{10.18}$$

The second integral is evaluated using the substitution $u = \sin x$

$$\int \frac{dx}{\cos^3 x} = \int \frac{\cos x\, dx}{\cos^4 x} = \int \frac{\cos x\, dx}{(1-\sin^2 x)^2} = \int \frac{du}{(1-u^2)^2} \tag{10.19}$$

and then employing the partial fraction expansion

$$\frac{1}{(1-u^2)^2} = \frac{1}{4} \left(\frac{1}{(u-1)^2} - \frac{1}{u-1} + \frac{1}{(u+1)^2} + \frac{1}{u+1} \right) \tag{10.20}$$

and simplifying the result to complete the proof. □

10.7. Entry 2.544.7.

$$\int \frac{\sin 3x}{\sin x} dx = x + \sin 2x \tag{10.21}$$

PROOF. The identity $\sin 3x = 3\sin x - 4\sin^3 x$ shows that

$$\int \frac{\sin 3x}{\sin x} dx = \int 3\, dx - 4 \int \sin^2 x\, dx. \tag{10.22}$$

The result now follows from the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. □

10.8. Entry 2.544.8.

$$\int \frac{\sin 3x}{\sin^2 x} dx = 3 \ln \tan \frac{x}{2} + 4\cos x \tag{10.23}$$

PROOF. The identity $\sin 3x = 3\sin x - 4\sin^3 x$ gives

$$\frac{\sin 3x}{\sin^2 x} = \frac{3}{\sin x} - 4\sin x. \tag{10.24}$$

The first integral was evaluated in the proof of (10.1) and the second one is elementary. □

10.9. Entry 2.544.9.

$$\int \frac{\sin 3x}{\sin^3 x} dx = -3\cot x - 4x \tag{10.25}$$

PROOF. The identity $\sin 3x = 3\sin x - 4\sin^3 x$ gives

$$\int \frac{\sin 3x}{\sin^3 x} dx = 3\int \frac{dx}{\sin^2 x} - 4\int 1\, dx. \tag{10.26}$$

The elementary result

$$\int \frac{dx}{\sin^2 x} = \int \text{cosec}^2 x\, dx = -\text{cot}\ x \tag{10.27}$$

completes the evaluation. □

11. Entries in Section 2.545

11.1. Entry 2.545.1.

$$\int \frac{\sin 3x}{\cos^n x}\, dx = \frac{4}{(n-3)\cos^{n-3} x} - \frac{1}{(n-1)\cos^{n-1} x} \tag{11.1}$$

For $n \neq 1, 3$.

PROOF. Use the identity $\sin 3x = 3\sin x - 4\sin^3 x = -\sin x + 4\sin x \cos^2 x$ gives

$$\int \frac{\sin 3x}{\cos^n x}\, dx = -\int \cos^{-n} x \sin x\, dx + 4\int \cos^{2-n} x \sin x\, dx. \tag{11.2}$$

The change of variables $u = \cos x$ confirms (11.1). □

11.2. Entry 2.545.2.

$$\int \frac{\sin 3x}{\cos x}\, dx = 2\sin^2 x + \ln\cos x \tag{11.3}$$

PROOF. Start with the identity $\sin 3x = 3\sin x - 4\sin^3 x$ to convert the integral into

$$\int \frac{\sin 3x}{\cos x}\, dx = 3\int \tan x\, dx - 4\int \frac{\sin^3 x}{\cos x}\, dx. \tag{11.4}$$

The first integral is elementary

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\ln\cos x \tag{11.5}$$

and to evaluate the second integral write it as

$$\begin{aligned} \int \frac{\sin^3 x}{\cos x}\, dx &= \int \frac{\sin x(1-\cos^2 x)}{\cos x}\, dx \\ &= \int \frac{\sin x}{\cos x}\, dx - \int \cos x \sin x\, dx \\ &= -\ln\cos x - \frac{1}{2}\sin^2 x. \end{aligned} \tag{11.6}$$

Replacing in (11.4) gives the result. □

11.3. Entry 2.545.3.

$$\int \frac{\sin 3x}{\cos^3 x}\, dx = -\frac{1}{2\cos^2 x} - 4 \ln \cos x \tag{11.7}$$

PROOF. The identity $\sin 3x = 3\sin x - 4\sin^3 x$ gives

$$\int \frac{\sin 3x}{\cos^3 x}\, dx = 3 \int \cos^{-3} x \, \sin x \, dx - 4 \int \cos^{-3} x \left(1 - \cos^2 x\right) \sin x \, dx. \tag{11.8}$$

The change of variables $u = \cos x$ evaluates these integrals and confirms (11.7). □

12. Entries in Section 2.546

12.1. Entry 2.546.1.

$$\int \frac{\cos 3x}{\sin^n x}\, dx = \frac{4}{(n-3)\sin^{n-3} x} - \frac{1}{(n-1)\sin^{n-1} x}, \tag{12.1}$$

for $n \neq 1,\ 3$.

PROOF. Start with the identity $\cos 3x = 4\cos^3 x - 3\sin x$ and write

$$\begin{aligned} \frac{\cos 3x}{\sin^n x} &= \frac{4\cos x(1-\sin^2 x) - 3\cos x}{\sin^n x} \\ &= \frac{\cos x}{\sin^n x} - \frac{4\cos x}{\sin^{n-2} x}. \end{aligned} \tag{12.2}$$

Now let $u = \sin x$ to obtain

$$\int \frac{\cos 3x}{\sin^n x}\, dx = \int u^{-n} du - 4 \int u^{2-n}\, du = \frac{u^{1-n}}{1-n} - \frac{4}{3-n} u^{3-n}. \tag{12.3}$$

This is the claim. □

12.2. Entry 2.546.2.

$$\int \frac{\cos 3x}{\sin x}\, dx = -2\sin^2 x + \ln \sin x \tag{12.4}$$

PROOF. The identity $\cos 3x = 4\cos^3 x - 3\cos x = \cos x - 4\cos x \sin^2 x$ gives

$$\int \frac{\cos 3x}{\sin x}\, dx = \int \frac{\cos x}{\sin x}\, dx - 4 \int \cos x \sin x \, dx. \tag{12.5}$$

Now make the change of variables $u = \sin x$ to obtain the result. □

12.3. Entry 2.546.3.

$$\int \frac{\cos 3x}{\sin^3 x}\, dx = -\frac{1}{2\sin^2 x} - 4 \ln \sin x \tag{12.6}$$

PROOF. The identity $\cos 3x = \cos x - 4\cos x \sin^2 x$ and the change of variables $u = \sin x$ gives the result. □

13. Entries in Section 2.547

13.1. Entry 2.547.1.

$$\int \frac{\sin nx}{\cos^p x}\,dx = 2\int \frac{\sin(n-1)x}{\cos^{p-1} x}\,dx - \int \frac{\sin(n-2)x}{\cos^p x}\,dx \tag{13.1}$$

Proof. The fact that the integrands match is the identity

$$\sin nx = 2\cos x \sin(n-1)x - \sin(n-2)x. \tag{13.2}$$

To prove it, use

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \tag{13.3}$$

with $A = nx$ and $B = (n-2)x$. This completes the proof. □

13.2. Entry 2.547.2.

$$\int \frac{\cos 3x}{\cos x}\,dx = \sin 2x - x \tag{13.4}$$

Proof. The identity

$$\cos 3x = 4\cos^3 x - 3\cos x \tag{13.5}$$

follows from the addition formula for trigonometric functions. Then

$$\int \frac{\cos 3x}{\cos x}\,dx = \int \left(4\cos^2 x - 3\right)\,dx. \tag{13.6}$$

The result now follows from the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$. □

13.3. Entry 2.547.3.

$$\int \frac{\cos 3x}{\cos^2 x}\,dx = 4\sin x - 3\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \tag{13.7}$$

Proof. The identity $\cos 3x = 4\cos^3 x - 3\cos x$ gives

$$\int \frac{\cos 3x}{\cos^2 x}\,dx = 4\int \cos x\,dx - 3\int \frac{dx}{\cos x}. \tag{13.8}$$

The first integral is elementary and the second one is evaluated by the Weiestrass substitution $t = \tan x/2$ from which

$$\cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad dx = \frac{2\,dt}{1+t^2}. \tag{13.9}$$

Then

$$\int \frac{dx}{\cos x} = \int \frac{2}{1-t^2} = \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right)\,dt, \tag{13.10}$$

so it follows that

$$\int \frac{dx}{\cos x} = \ln\left(\frac{1+t}{1-t}\right) = \ln\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) \tag{13.11}$$

and this reduces to the stated expression by the addition theorem for tangent. □

13.4. Entry 2.547.4.

$$\int \frac{\cos 3x}{\cos^3 x}\,dx = 4x - 3\tan x \tag{13.12}$$

PROOF. The identity $\cos 3x = 4\cos^3 x - 3\cos x$ yields

$$\int \frac{\cos 3x}{\cos^3 x}\,dx = 4\int 1\,dx - 3\int \frac{dx}{\cos^2 x} = 4x - 3\tan x \tag{13.13}$$

as claimed. □

14. A non-elementary evaluation

The goal of this section is to present an expression for

$$I_{p,q}(x) = \int \sin^p x \sin(qx)\,dx \tag{14.1}$$

as a finite sum involving the integrals

$$J_a(x) = \int \sin^a x\,dx. \tag{14.2}$$

The special case

$$I_{p,2n+1}(x) = \int \sin^p x \sin(2n+1)x\,dx \tag{14.3}$$

appearing as entry **2.535.2** is discussed in detail.

The first step is a reduction formula transforming the term $\sin ax$ into a $\cos(a-1)x$ form.

LEMMA 14.1. *The identity*

$$\int \sin^p x\,\sin ax\,dx = \frac{1}{p+a}\left\{-\sin^p x\cos ax + p\int \sin^{p-1} x\cos(a-1)x\,dx\right\} \tag{14.4}$$

holds.

PROOF. Integrate by parts to obtain

$$\int \sin^p x\,\sin ax\,dx = -\frac{1}{a}\sin^p x\cos ax + \frac{p}{a}\int \sin^{p-1} x\cos x\cos ax\,dx. \tag{14.5}$$

Now use $\cos ax\cos x = \cos(a-1)x - \sin ax\sin x$ to obtain the result. □

The second step proves a similar reduction converting $\cos ax$ back into a sine term.

LEMMA 14.2. *The identity*

$$\int \sin^p x\,\cos ax\,dx = \frac{1}{p+a}\left\{\sin^p x\sin ax - p\int \sin^{p-1} x\sin(a-1)x\,dx\right\} \tag{14.6}$$

holds.

Proof. Integration by parts gives

$$\int \sin^p x \, \cos ax \, dx = \frac{1}{a} \sin^p x \, \sin ax - \frac{p}{a} \int \sin^{p-1} x \cos x \, \sin ax \, dx. \tag{14.7}$$

Now use $\cos x \sin ax = \sin((a-1)x) + \cos ax \sin x$ to obtain the result. □

Now use Lemma 14.2 in Lemma 14.1, with $a = 2n+1$, to produce, with the notation in (14.3),

$$\begin{aligned} I_{p,2n+1} &= -\frac{1}{p+2n+1} \sin^p x \cos(2n+1)x + \frac{p}{(p+2n+1)(p+2n-1)} \sin^{p-1} x \sin 2nx \\ &\quad - \frac{p(p-1)}{(p+2n+1)(p+2n-1)} I_{p-2,2n-1}. \end{aligned}$$

A one step iteration of this recurrence produces

$$\begin{aligned} I_{p,2n+1} &= -\frac{1}{p+2n+1} \sin^p x \cos(2n+1)x \\ &\quad + \frac{p}{(p+2n+1)(p+2n-1)} \sin^{p-1} x \sin(2nx) \\ &\quad + \frac{p(p-1)}{(p+2n+1)(p+2n-1)(p+2n-3)} \sin^{p-2} x \cos(2n-1)x \\ &\quad - \frac{p(p-1)(p-2)}{(p+2n+1)(p+2n-1)(p+2n-3)(p+2n-5)} \sin^{p-3} x \sin((2n-2)x) \\ &\quad + \frac{p(p-1)(p-2)(p-3)}{(p+2n+1)(p+2n-1)(p+2n-3)(p+2n-5)} I_{p-4,2n-3}. \end{aligned} \tag{14.8}$$

It is convenient to write this recurrences using the Pochhammer symbol defined, for $x \in \mathbb{R}$ and $\ell = 0, 1, 2, \cdots$, by

$$(x)_\ell = \begin{cases} x(x+1)(x+2)\cdots(x+\ell-1) & \text{for } \ell = 1, 2, \cdots \\ 1, & \text{for } \ell = 0. \end{cases} \tag{14.9}$$

Introduce the notation

$$T = -\frac{p+2n+1}{2}, \quad Q = -p. \tag{14.10}$$

Then (14.8) can be written as

$$\begin{aligned} I_{p,2n+1} = -\frac{(Q)_0}{2(T)_1} \sin^p x \cos(2n+1)x \\ - \frac{(Q)_1}{2^2(T)_2} \sin^{p-1} x \sin(2nx) - \frac{(Q)_2}{2^2(T)_2} I_{p-2,2n-1}. \end{aligned} \tag{14.11}$$

Iterating this procedure leads to the recurrence

$$
\begin{aligned}
(14.12) \qquad I_{p,2n+1} &= \sum_{k=0}^{2j-1} \frac{(-1)^k (Q)_{2k}}{2^{2k+1}(T)_{2k+1}} \sin^{p-2k} x \cos[2(n-k)+1]x \\
&\quad - \sum_{k=0}^{2j-1} \frac{(-1)^k (Q)_{2k+1}}{2^{2(k+1)}(T)_{2(k+1)}} \sin^{p-2k-1} x \sin[2(n-k)x] \\
&\quad + \frac{(Q)_{4j}}{2^{4j}(T)_{4j}} I_{p-4j,2n-4j+1}.
\end{aligned}
$$

The last reduction step is to write the quotients of Pochhammer symbols. In detail

$$
\begin{aligned}
(14.13) \qquad \frac{(Q)_{2k}}{(T)_{2k+1}} &= -\frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \frac{\Gamma\left(\frac{p+1}{2}+n-2k\right)}{\Gamma(p+1-2k)} \\
\frac{(Q)_{2k+1}}{(T)_{2k+2}} &= -\frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{\Gamma(p-2k)}.
\end{aligned}
$$

This leads to the proof of Entry **2.535.2**:

$$
\begin{aligned}
(14.14) \qquad & \int \sin^p x \, \sin(2n+1)x \, dx = \\
& \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^{k-1}\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)} \sin^{p-2k} x \, \cos(2n-2k+1)x \right. \right. \\
& \qquad \left. + (-1)^k \frac{\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\Gamma(p-2k)} \sin^{p-2k-1} x \, \sin(2n-2k)x \right] \\
& \qquad \left. + \frac{(-1)^n \Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\Gamma(p-2n+1)} \int \sin^{p-2n+1} x \, dx \right\}
\end{aligned}
$$

This entry also includes and alternative expression for this integral based on the fact that $\sin(2n+1)x$ can be expressed in terms of the Chebyshev polynomial of the second kind. Details will appear elsewhere.

There is a small list of entries with similar statements as the example discussed above. The proofs are similar and are left to the reader.

Entry 2.535.3.

$$
\begin{aligned}
& \int \sin^p x \, \sin 2nx \, dx = 2n \left\{ \frac{\sin^{p+2} x}{p+2} + \right. \\
& = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^{k-1}\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)} \sin^{p-2k} x \, \cos(2n-2k)x \right. \right. \\
& \qquad \left. - (-1)^k \frac{\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\Gamma(p-2k)} \sin^{p-2k-1} x \, \sin(2n-2k-1)x \right\}
\end{aligned}
$$

for p not equal to $-2, -4, \ldots, -2n$.

Entry 2.536.2.

$$\int \sin^p x\cos(2n{+}1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left\{\sum_{k=0}^{n-1}\left[\frac{(-1)^k\Gamma\left(\frac{p+1}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)}\sin^{p-2k} x\,\sin(2n-2k+1)x\right.\right.$$
$$\left.+\frac{(-1)^k\Gamma\left(\frac{p-1}{2}+n-2k\right)}{2^{2k+2}\Gamma(p-2k)}\sin^{p-2k-1} x\cos(2n-2k)x\right]$$
$$\left.+\frac{(-1)^n\Gamma\left(\frac{p+3}{2}-n\right)}{2^{2n}\Gamma(p-2n+1)}\int \sin^{p-2n} x\cos x\,dx\right\}$$

for p not equal to $-3,\ -5,\ \ldots, -(2n+1)$.

Entry 2.536.3.

$$\int \sin^p x\cos 2nx\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)}\left\{\sum_{k=0}^{n-1}\left[\frac{(-1)^k\Gamma\left(\frac{p}{2}+n-2k\right)}{2^{2k+1}\Gamma(p-2k+1)}\sin^{p-2k} x\sin(2n-2k)x\right.\right.$$
$$\left.\frac{(-1)^k\Gamma\left(\frac{p}{2}+n-2k-1\right)}{2^{2k+2}\Gamma(p-2k)}\sin^{p-2k-1} x\cos(2n-2k-1)x\right]$$
$$\left.+\frac{(-1)^n\Gamma\left(\frac{p}{2}-n+1\right)}{2^{2n}\Gamma(p-2n+1)}\int \sin^{p-2n} x\,dx\right\}$$

Entry 2.537.2.

$$\int \cos^p x\sin(2n{+}1)x\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)}\left\{-\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{2k+1}\Gamma(p-2k+1)}\cos^{p-k} x\cos(2n-k+1)x\right.$$
$$\left.+\frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\Gamma(p-n+1)}\int \cos^{p-n} x\sin(n+1)x\,dx\right\}$$

for p not equal to $-3,\ -5,\ \ldots, -(2n+1)$.

Entry 2.537.3.

$$\int \cos^p x\sin 2nx\,dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)}\left\{-\sum_{k=0}^{n-1}\frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\Gamma(p-k+1)}\cos^{p-k} x\cos(2n-k)x\right.$$
$$\left.+\frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\Gamma(p-n+1)}\int \cos^{p-n} x\ \sin nx\,dx\right\}$$

Entry 2.538.2.

$$\int \cos^p x \, \cos(2n+1)x \, dx \times \int \cos^{2k+p+1} x \, dx \Big\}$$

$$= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2}+n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2}+n-k\right)}{2^{k+1}\Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k+1)x \right.$$

$$\left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\Gamma(p-n+1)} \int \cos^{p-n} x \cos(n+1)x \, dx \right\}$$

Entry 2.538.3.

$$\int \cos^p x \, \cos 2nx \, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+n+1\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2}+n-k\right)}{2^{k+1}\Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k)x \right.$$

$$\left. + \frac{\Gamma\left(\frac{p}{2}+1\right)}{2^n\Gamma(p-n+1)} \int \cos^{p-n} x \, \cos nx \, dx \right\}$$

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