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Sums of arctangents and some formulas of Ramanujan

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Abstract. We present diverse methods to evaluate arctangent and related sums.

1. Introduction

The evaluation of arctangent sums of the form

$$\sum_{k=1}^{\infty} \tan^{-1} h(k) \tag{1.1}$$

for a rational function h reappear in the literature from time to time. For instance the evaluation of

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} = \frac{3\pi}{4} \tag{1.2}$$

was proposed by Anglesio [**1**] in 1993. This is a classical problem that appears in [**7, 9, 13**], among other places. Similarly the evaluation of

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^2} = \tan^{-1} \frac{\tan(\pi/\sqrt{2}) - \tanh(\pi/\sqrt{2})}{\tan(\pi/\sqrt{2}) + \tanh(\pi/\sqrt{2})} \tag{1.3}$$

was proposed by Chapman [**6**] in 1990. This was solved by Sarkar [**15**] using the techniques described in Section 3.

The goal of this paper is to discuss the evaluation of these sums. Throughout $\tan^{-1} x$ will always denote the principal value.

We make use of the addition formulas for $\tan^{-1} x$:

$$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x+y}{1-xy} & \text{if } xy < 1, \\ \tan^{-1} \frac{x+y}{1-xy} + \pi \,\text{sign}\, x & \text{if } xy > 1, \end{cases} \tag{1.4}$$

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and

$$\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2} \text{ sign } x. \tag{1.5}$$

2. The method of telescoping

The closed-form evaluation of a finite sum

$$S(n) := \sum_{k=1}^{n} a_k$$

is elementary if one can find a sequence $\{b_k\}$ such that

$$a_k = b_k - b_{k-1}.$$

Then the sum $S(n)$ telescopes, i.e.,

$$S(n) := \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k - b_{k-1} = b_n - b_0.$$

This method can be extended to situations in which the telescoping nature of a_k is hidden by a function.

THEOREM 2.1. *Let f be of fixed sign and define h by*

$$h(x) = \frac{f(x+1) - f(x)}{1 + f(x+1)f(x)}. \tag{2.1}$$

Then

$$\sum_{k=1}^{n} tan^{-1} h(k) = tan^{-1} f(n+1) - tan^{-1} f(1). \tag{2.2}$$

In particular, if f has a limit at ∞ (including the possibility of $f(\infty) = \infty$), then

$$\sum_{k=1}^{\infty} tan^{-1} h(k) = tan^{-1} f(\infty) - tan^{-1} f(1). \tag{2.3}$$

PROOF. Since

$$\tan^{-1} h(k) = \tan^{-1} f(k+1) - \tan^{-1} f(k),$$

(2.2) follows by telescoping. □

Note. The hypothesis on the sign of f is included in order to avoid the case $xy > 1$ in (1.4). In general, (2.2) has to be replaced by

$$\sum_{k=1}^{n} \tan^{-1} h(k) = \tan^{-1} f(n) - \tan^{-1} f(1) + \pi \sum \text{sign } f(k), \tag{2.4}$$

where the sum is taken over all k between 1 and n for which $f(k)f(k+1) < -1$. Thus (2.2) is always correct up to an integral multiple of π. The restrictions on the parameters in the examples described below have the intent of keeping $f(k),\ k \in \mathbb{N}$ of fixed sign.

EXAMPLE 2.1. Let $f(x) = ax + b$, where $a,\ b$ are such that $f(x) \geqslant 0$ for $x \geqslant 1$. Then

$$h(x) = \frac{a}{a^2x^2 + a(a+2b)x + (1+ab+b^2)}, \tag{2.5}$$

and (2.3) yields

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{a}{a^2k^2 + a(a+2b)k + (1+ab+b^2)} = \frac{\pi}{2} - \tan^{-1}(a+b). \tag{2.6}$$

The special case $a = 1$, $b = 0$ gives $f(x) = x$ and $h(x) = 1/(x^2 + x + 1)$, resulting in the sum

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^2 + k + 1} = \frac{\pi}{4}. \tag{2.7}$$

For $a = 2$, $b = 0$, we get $f(x) = 2x$ and $h(x) = 2/(2x + 1)^2$, so that

$$\sum_{k=0}^{\infty} \tan^{-1} \frac{2}{(2k+1)^2} = \frac{\pi}{2}. \tag{2.8}$$

Differentiating (2.6) with respect to a yields

$$\sum_{k=1}^{\infty} \frac{p_{a,b}(k)}{q_{a,b}(k)} = \frac{1}{1 + (a+b)^2}, \tag{2.9}$$

where

$$p_{a,b}(k) = a^2k^2 + a^2k - (1+b^2)$$

and

$$\begin{aligned} q_{a,b}(k) = {} & a^4k^4 + 2a^3(a+2b)k^3 + a^2(2 + a^2 + 6ab + 6b^2)k^2 + \\ & 2a(a+2b)(1+ab+b^2)k + (1+b^2)(1+a^2+2ab+b^2). \end{aligned}$$

The particular cases $a = 1$, $b = 0$ and $a = 1/2$, $b = 1/3$ give

$$\sum_{k=1}^{\infty} \frac{k^2 + k - 1}{k^4 + 2k^3 + 3k^2 + 2k + 2} = \frac{1}{2}$$

and

$$\sum_{k=1}^{\infty} \frac{9k^2 + 9k - 40}{81k^4 + 378k^3 + 1269k^2 + 1932k + 2440} = \frac{1}{61},$$

respectively.

EXAMPLE 2.2. This example considers the quadratic function $f(x) = ax^2 + bx + c$ under the assumption that $f(k),\ k \in \mathbb{N}$ has fixed sign. This happens when $b^2 - 4ac \leqslant 0$.

Define

$$\begin{aligned} a_0 &:= 1 + ac + bc + c^2, \\ a_1 &:= ab + b^2 + 2ac + 2bc, \\ a_2 &:= a^2 + 3ab + b^2 + 2ac, \\ a_3 &:= 2a(a+b), \\ a_4 &:= a^2. \end{aligned}$$

Then,

$$h(x) = \frac{2ax + a + b}{a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0} \tag{2.10}$$

and thus,

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2ak + a + b}{a_4k^4 + a_3k^3 + a_2k^2 + a_1k + a_0} = \frac{\pi}{2} - \tan^{-1}(a+b+c).$$

The special cases $b = -a,\ c = a/2$ and $b = -a,\ c = 0$ yield

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{8ak}{4a^2k^4 + (a^2+4)} = \frac{\pi}{2} - \tan^{-1} \frac{a}{2}$$

and

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2ak}{a^2k^4 - a^2k^2 + 1} = \frac{\pi}{2}, \tag{2.11}$$

respectively. Note that the last sum is independent of a.

Additional examples can be given by telescoping twice (or even more). For example, if f and h be related by

$$h(x) = \frac{f(x+1) - f(x-1)}{1 + f(x+1)f(x-1)}, \tag{2.12}$$

then

$$\sum_{k=1}^{n} \tan^{-1} h(k) = \tan^{-1} f(n+1) - \tan^{-1} f(1) + \tan^{-1} f(n) - \tan^{-1} f(0).$$

In particular,

$$\sum_{k=1}^{\infty} \tan^{-1} h(k) = 2\tan^{-1} f(\infty) - \tan^{-1} f(1) - \tan^{-1} f(0). \tag{2.13}$$

Indeed, the relation (2.12) shows that

$$\tan^{-1} h(k) = \tan^{-1} f(k+1) - \tan^{-1} f(k-1),$$

so

$$\sum_{k=1}^{n} \tan^{-1} h(k) =$$

$$\begin{aligned} &= \sum_{k=1}^{n} \left[\tan^{-1} f(k+1) - \tan^{-1} f(k-1)\right] \\ &= \sum_{k=1}^{n} \left[\tan^{-1} f(k+1) - \tan^{-1} f(k)\right] + \sum_{k=1}^{n} \left[\tan^{-1} f(k) - \tan^{-1} f(k-1)\right] \\ &= \tan^{-1} f(n+1) - \tan^{-1} f(1) + \tan^{-1} f(n) - \tan^{-1} f(0). \end{aligned}$$

EXAMPLE 2.3. The evaluation

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} = \frac{3\pi}{4} \tag{2.14}$$

corresponds to $f(k) = k$ so that $h(k) = 2/k^2$. This is the problem proposed by Anglesio [**1**].

EXAMPLE 2.4. Take $f(k) = -2/k^2$ so that $h(k) = 8k/(k^4 - 2k^2 + 5)$. It follows that

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{8k}{k^4 - 2k^2 + 5} = \pi - \tan^{-1} \frac{1}{2}.$$

This sum is part b) of the problem proposed in [**1**].

EXAMPLE 2.5. Take $f(k) = -a/(k^2+1)$. Then $h(k) = 4ak/(k^4 + a^2 + 4)$, so that

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4ak}{k^4 + a^2 + 4} = \tan^{-1} \frac{a}{2} + \tan^{-1} a. \tag{2.15}$$

The case $a = 1$ yields

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{4k}{k^4 + 5} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2}. \tag{2.16}$$

Differentiating (2.15) with respect to a gives

$$\sum_{k=1}^{\infty} \frac{4k(k^4 + 4 - a^2)}{k^8 + 2(a^2+4)k^4 + 16a^2k^2 + (a^4 + 8a^2 + 16)} = \frac{3(a^2+2)}{(a^2+1)(a^2+4)}.$$

The special case $a = 0$ yields

$$\sum_{k=1}^{\infty} \frac{k}{k^4 + 4} = \frac{3}{8}. \tag{2.17}$$

An interesting problem is to find a closed form for f given the function h in (2.1) or (2.12). Unfortunately this is not possible in general. Moreover, these equations might have more than one solution: both $f(x) = 2x + 1$ and $f(x) = -x/(x+1)$ yield $h(x) = -1/2x^2$ in (2.1). The method of undetermined coefficients can sometimes be used to find the function f. For instance, in Example 2.3 we need to solve the functional equation

$$2\left[1 + f(x-1)f(x+1)\right] = x^2\left[f(x+1) - f(x-1)\right]. \tag{2.18}$$

A polynomial solution of (2.18) must have degree at most 2 and trying $f(x) = ax^2 + bx + c$ yields the solution $f(x) = x$.

3. The method of zeros

A different technique for the evaluation of arctangent sums is based on the factorization of the product

$$p_n := \prod_{k=1}^{n} (a_k + ib_k) \tag{3.1}$$

with a_k, $b_k \in \mathbb{R}$. The argument of p_n is given by

$$\text{Arg}(p_n) = \sum_{k=1}^{n} \tan^{-1} \frac{b_k}{a_k}.$$

Example 3.1. Let

$$p_n(z) = \prod_{k=1}^{n} (z - z_k) \tag{3.2}$$

be a polynomial with real coefficients. Then

$$\text{Arg}(p_n(z)) = \sum_{k=1}^{n} \tan^{-1} \frac{x - x_k}{y - y_k}. \tag{3.3}$$

The special case $p_n(z) = z^n - 1$ has roots at $z_k = \cos(2\pi k/n) + i \sin(2\pi k/n)$, so we obtain

$$\text{Arg}(z^n - 1) = \sum_{k=1}^{n} \tan^{-1} \frac{x - \cos(2\pi k/n)}{y - \sin(2\pi k/n)} \tag{3.4}$$

up to an integral multiple of π.

Example 3.2. The classical factorization

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2} \right) \tag{3.5}$$

yields the evaluation

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2xy}{k^2 - x^2 + y^2} = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{\tanh \pi y}{\tan \pi x}. \tag{3.6}$$

In particular, $x = y$ yields

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2x^2}{k^2} = \frac{\pi}{4} - \tan^{-1} \frac{\tanh \pi x}{\tan \pi x}, \tag{3.7}$$

$x = y = 1/\sqrt{2}$ gives

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^2} = \frac{\pi}{4} - \tan^{-1} \frac{\tanh (\pi/\sqrt{2})}{\tan(\pi/\sqrt{2})} \tag{3.8}$$

(which corresponds to (1.3)), and $x = y = 1/2$ yields

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{2k^2} = \frac{\pi}{4}. \tag{3.9}$$

Differentiating (3.7) gives

$$\sum_{k=1}^{\infty} \frac{k^2}{k^4 + 4x^4} = \frac{\pi}{4x} \frac{\sin 2\pi x - \sinh 2\pi x}{\cos 2\pi x - \cosh 2\pi x}. \tag{3.10}$$

In particular, $x = 1$ yields

$$\sum_{k=1}^{\infty} \frac{k^2}{k^4 + 4} = \frac{\pi}{4} \coth \pi. \tag{3.11}$$

The identity (3.10) is comparable to Ramanujan's evaluation

$$\sum_{k=1}^{\infty} \frac{k^2}{k^4 + x^2k^2 + x^4} = \frac{\pi}{2x\sqrt{3}} \frac{\sinh \pi x\sqrt{3} - \sqrt{3}\sin \pi x}{\cosh \pi x\sqrt{3} - \cos \pi x} \tag{3.12}$$

discussed in [**3**], Entry 4 of Chapter 14.

Glasser and Klamkin [**10**] present other examples of this technique.

4. A functional equation

The table of sums and integrals [**11**] contains a small number of examples of finite sums that involve trigonometric functions of multiple angles. In Section 1.36 we find

$$\sum_{k=1}^{n} 2^{2k} \sin^4 \frac{x}{2^k} = 2^{2n} \sin^2 \frac{x}{2^n} - \sin^2 x, \tag{4.1}$$

and

$$\sum_{k=1}^{n} \frac{1}{2^{2k}} \sec^2 \frac{x}{2^k} = \operatorname{cosec}^2 x - \frac{1}{2^{2n}} \operatorname{cosec}^2 \frac{x}{2^n}, \tag{4.2}$$

and Section 1.37 consists entirely of the two sums

$$\sum_{k=0}^{n} \frac{1}{2^k} \tan \frac{x}{2^k} = \frac{1}{2^n} \cot \frac{x}{2^n} - 2 \cot 2x \tag{4.3}$$

and

$$\sum_{k=0}^{n} \frac{1}{2^{2k}} \tan^2 \frac{x}{2^k} = \frac{2^{2n+2} - 1}{3 \cdot 2^{2n-1}} + 4 \cot^2 2x - \frac{1}{2^{2n}} \cot^2 \frac{x}{2^n}. \tag{4.4}$$

In this section we present a systematic procedure to analyze these sums.

THEOREM 4.1. *Let*

$$F(x) = \sum_{k=1}^{\infty} f(x,k) \quad \textit{and} \quad G(x) = \sum_{k=1}^{\infty} (-1)^k f(x,k). \tag{4.5}$$

Suppose $f(x, 2k) = \nu f(\lambda(x), k)$ *for some* $\nu \in \mathbb{R}$ *and a function* $\lambda : \mathbb{R} \to \mathbb{R}$. *Then*

$$F(x) \quad = \quad (2\nu)^n F(\lambda^{[n]}(x)) - \sum_{j=0}^{n-1} (2\nu)^j G(\lambda^{[j]}(x)), \tag{4.6}$$

where $\lambda^{[n]}$ *denotes the composition of* λ *with itself* n *times.*

PROOF. Observe that

$$F(x) + G(x) = 2\sum_{k=1}^{\infty} f(x, 2k) = 2\nu \sum_{k=1}^{\infty} f(\lambda(x), k) = 2\nu F(\lambda(x)).$$

Repeat this argument to obtain the result. □

EXAMPLE 4.1. Let $f(x, k) = 1/(x^2 + k^2)$, so that $\nu = 1/4$ and $\lambda(x) = x/2$. Since

$$F(x) = \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} = \frac{\pi x \text{ coth } \pi x - 1}{2x^2}$$

and

$$G(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 + k^2} = \frac{\pi x \text{ csch } \pi x - 1}{2x^2},$$

(4.6) yields, upon letting $n \to \infty$,

$$\sum_{j=0}^{\infty} \frac{x}{\sinh 2^{-j} x} - 2^j \quad = \quad 1 - \frac{x}{\tanh x}. \tag{4.7}$$

Now replace x by $\ln t$, differentiate with respect to t, and set $t = e$ to produce

$$\sum_{j=0}^{\infty} \frac{2^j - \text{ coth } 2^{-j}}{2^j \text{ sinh } 2^{-j}} \quad = \quad \frac{1 + 4e^2 - e^4}{1 - 2e^2 + e^4}. \tag{4.8}$$

If we go back to (4.7), replace x by $\ln t$, differentiate with respect to t, set $t = ae$, differentiate with respect to a, and set $a = e$, we get

$$\sum_{j=0}^{\infty} \frac{2 - 2^{2j} + \text{ csch}^2 2^{-j} - \text{sech}^2 2^{-j}}{2^{2j} \text{ sinh} 2^{1-j}} \quad = \quad \frac{e^{12} - 17e^8 - 17e^4 + 1}{e^{12} - 3e^8 + 3e^4 - 1}. \tag{4.9}$$

COROLLARY 4.1. *Let*

$$F(x) = \sum_{k=1}^{\infty} f\left(\frac{x}{k}\right) \quad \textit{and} \quad G(x) = \sum_{k=1}^{\infty} (-1)^k f\left(\frac{x}{k}\right). \tag{4.10}$$

Then, for any $n \in \mathbb{N}$,

$$F(x) \quad = \quad 2^{-n} F(2^n x) + \sum_{k=1}^{n} 2^{-k} G(2^k x). \tag{4.11}$$

In particular, if F is bounded, then

$$F(x) = \sum_{k=1}^{\infty} 2^{-k} G(2^k x). \tag{4.12}$$

Proof. The function $f(x/k)$ satisfies the conditions of Theorem 4.1 with $\nu = 1$ and $\lambda(x) = x/2$. Thus

$$F(x) = 2^n F(x/2^n) - \sum_{j=1}^{n-1} 2^j G(x/2^j).$$

Now replace x by $x/2^n$ to obtain (4.11). Finally, let $n \to \infty$ to obtain (4.12). □

The key to the proof of Theorem 4.1 is the identity $F(x) + G(x) = 2\nu F(\lambda(x))$. We next present an extension of this result.

Theorem 4.2. *Let F, G be functions that satisfy*

$$F(x) = r_1 F(m_1 x) + r_2 G(m_2 x) \tag{4.13}$$

for parameters r_1, r_2, m_1, m_2. Then

$$r_2 \sum_{k=1}^{n} r_1^{k-1} G(m_1^{k-1} m_2 x) = F(x) - r_1^n F(m_1^n x). \tag{4.14}$$

Proof. Replace x by $m_1 x$ in (4.13) to produce

$$F(m_1 x) = r_1 F(m_1^2 x) + r_2 G(m_2 m_1 x),$$

which, when combined with (4.13), gives

$$F(x) = r_1^2 F(m_1^2 x) + r_1 r_2 G(m_1 m_2 x) + r_2 G(m_2 x).$$

Formula (4.14) follows by induction. □

We now present two examples that illustrate Theorem 4.2. These sums appear as entries in Ramanujan's Notebooks.

Example 4.2. The identity

$$\cot x = \frac{1}{2} \cot \frac{x}{2} - \frac{1}{2} \tan \frac{x}{2} \tag{4.15}$$

shows that $F(x) = \cot x$, $G(x) = \tan x$ satisfy (4.13) with $r_1 = 1/2$, $r_2 = -1/2$, and $m_1 = m_2 = 1/2$. We conclude that

$$\sum_{k=1}^{n} 2^{-k} \tan \frac{x}{2^k} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x. \tag{4.16}$$

This is (4.3). It also appears as Entry 24, page 364, of Ramanujan's Third Notebook as described in Berndt [**4**, page 396]. Similarly, the identity

$$\sin^2(2x) = 4 \sin^2 x - 4 \sin^4 x \tag{4.17}$$

yields (4.1). The reader is invited to produce proofs of (4.2) and (4.4) in the style presented here.

Example 4.3. The identity

$$\cot x = \cot\frac{x}{2} - \csc x \tag{4.18}$$

satisfies (4.13) with $F(x) = \cot x$, $G(x) = \csc x$ and parameters $r_1 = 1$, $r_2 = -1$, $m_1 = 1/2$, $m_2 = 1$. We obtain

$$\sum_{k=1}^{n} \csc\frac{x}{2^{k-1}} = \cot\frac{x}{2^n} - \cot x. \tag{4.19}$$

This appears in the proof of Entry 27 of Ramanujan's Third Notebook in Berndt [**4**, page 398].

Example 4.4. The application of Theorem 4.1 or Corollary 4.1 requires an analytic expression for F and G. One source of such expressions is Jolley [**12**]. Indeed, entries 578 and 579 are

$$\sum_{k=1}^{\infty} \tan^{-1}\frac{2x^2}{k^2} = \frac{\pi}{4} - \tan^{-1}\frac{\tanh \pi x}{\tan \pi x} \tag{4.20}$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1}\tan^{-1}\frac{2x^2}{k^2} = -\frac{\pi}{4} + \tan^{-1}\frac{\sinh \pi x}{\sin \pi x}. \tag{4.21}$$

These results also appear in [**5**, page 314]. Applying one step of Proposition 4.1 we conclude that

$$2\tan^{-1}\frac{\tanh x}{\tan x} = \tan^{-1}\frac{\tanh 2x}{\tan 2x} + \tan^{-1}\frac{\sinh 2x}{\sin 2x}. \tag{4.22}$$

We also obtain

$$\sum_{k=1}^{n} 2^{-k}\tan^{-1}\frac{\sinh 2^k x}{\sin 2^k x} = \tan^{-1}\frac{\tanh x}{\tan x} - 2^{-n}\tan^{-1}\frac{\tanh 2^n x}{\tan 2^n x}, \tag{4.23}$$

and by the boundedness of $\tan^{-1} x$ conclude that

$$\sum_{k=1}^{\infty} 2^{-k}\tan^{-1}\frac{\sinh 2^k x}{\sin 2^k x} = \tan^{-1}\frac{\tanh x}{\tan x}. \tag{4.24}$$

Differentiating (4.23) gives

$$2\sum_{k=1}^{n} \frac{\cos 2^k x \sinh 2^k x - \cosh 2^k x \sin 2^k x}{\cos 2^{k+1} x - \cosh 2^{k+1} x}$$

$$= -\frac{\sin 2x - \sinh 2x}{\cos 2x - \cosh 2x} + \frac{\sin 2^{n+1} x - \sinh 2^{n+1} x}{\cos 2^{n+1} x - \cosh 2^{n+1} x}.$$

Letting $n \to \infty$ and using the identity

$$\cos 2^{k+1} x - \cosh 2^{k+1} x = -2\left(\sin^2 2^k x + \sinh^2 2^k x\right) \tag{4.25}$$

yields

$$\sum_{k=1}^{\infty} \frac{\cosh 2^k x \sin 2^k x - \sinh 2^k x \cos 2^k x}{\sin^2 2^k x + \sinh^2 2^k x} = \frac{\operatorname{sech}^2 x \tan x - \tanh x \sec^2 x}{\tan^2 x + \tanh^2 x} + \operatorname{sign} x.$$

For example, $x = \pi$ gives

$$\sum_{k=1}^{\infty} \operatorname{csch} 2^k \pi = \coth \pi - 1. \tag{4.26}$$

5. A dynamical system

In this section we describe a dynamical system involving arctangent sums. Define

$$x_n = \tan \sum_{k=1}^{n} \tan^{-1} k \quad \text{and} \quad y_n = \tan \sum_{k=1}^{n} \tan^{-1} \frac{1}{k}.$$

Then $x_1 = y_1 = 1$ and

$$x_n = \frac{x_{n-1} + n}{1 - n x_{n-1}} \quad \text{and} \quad y_n = \frac{n y_{n-1} + 1}{n - y_{n-1}}.$$

Proposition 5.1. *Let $n \in \mathbb{N}$. Then*

$$x_n = \begin{cases} -y_n & \text{if } n \text{ is even} \\ 1/y_n & \text{if } n \text{ is odd} \end{cases} \tag{5.1}$$

that is

$$\tan \sum_{k=1}^{n} \tan^{-1} k = -\tan \sum_{k=1}^{n} \tan^{-1} \frac{1}{k} \tag{5.2}$$

if n is even and

$$\tan \sum_{k=1}^{n} \tan^{-1} k = \operatorname{cotg} \sum_{k=1}^{n} \tan^{-1} \frac{1}{k} \tag{5.3}$$

if n is odd.

Proof. The recurrence formulas for x_n and y_n can be used to prove the result directly. A pure trigonometric proof is presented next. If n is even then

$$\begin{aligned} \tan \sum_{k=1}^{2m} \tan^{-1} k + \tan \sum_{k=1}^{2m} \tan^{-1} \frac{1}{k} &= \tan \sum_{k=1}^{2m} \tan^{-1} k + \tan \sum_{k=1}^{2m} \left(\pi/2 - \tan^{-1} k\right) \\ &= \tan \sum_{k=1}^{2m} \tan^{-1} k + \tan \left(\pi m - \sum_{k=1}^{2m} \tan^{-1} k\right) \\ &= 0. \end{aligned}$$

A similar argument holds for n odd. □

This dynamical system suggests many interesting questions. We conclude by proposing one of them: *Observe that $x_3 = 0$. Does this ever happen again?*

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