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Networks in Ponomarev-systems

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ABSTRACT. Let $(f, M, X, \mathcal{P})$ be a *Ponomarev*-system. We prove that f is a compact-covering mapping iff $\mathcal{P}$ is a strong k-network of X. Furthermore, f is a compact-covering, s-mapping iff $\mathcal{P}$ is a point-countable cfp-network (or point-countable strong k-network) of X. As an application of these results, a point-countable cover of a space is a strong k-network iff it is a cfp-network, where "point-countable" can not be omitted.

1. Introduction

In 1960, Ponomarev [**8**] proved that every first countable space can be characterize as an open image of a subspace of a Baire's zero-dimensional space. Recently Lin [**3**] generalized the "Ponomarev's method" to established a system $(f, M, X, \mathcal{P})$, which is called a *Ponomarev*-system [**3, 5, 9**]. The following results have be obtained [**3, 5, 9**].

THEOREM 1.1. *The following are hold for a Ponomarev-system* $(f, M, X, \mathcal{P})$.
(1) If $\mathcal{P}$ *is a point-finite (resp. point-countable) network of* X*, then* f *is a compact mapping (resp.* s*-mapping).*
(2) If $\mathcal{P}$ *is a point-countable* cfp*-network of* X*, then* f *is a compact-covering,* s*-mapping.*

Take Theorem 1.1 into account, the following questions naturally arise.

QUESTION 1.2. Let $(f, M, X, \mathcal{P})$ be a *Ponomarev*-system.
(1) Can implications in Theorem 1.1 be reversed? Furthermore, can implications in Theorem 1.1(2) be reversed if both "point-countable" and "s-" are omitted?
(2) Can both "point-countable" and "s-" in Theorem 1.1(2) be omitted?
(3) More precisely, what is the necessary-and-sufficient condition such that f is a compact-covering mapping?

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In this paper, we give a necessary-and-sufficient conditions such that f is a compact-covering mapping. Furthermore, we affirmatively and negatively answer Question 1.2(1) and Question 1.2(2) respectively. As an application of these results, a point-countable cover of a space is a strong k-network iff it is a cfp-network, where "point-countable" can not be omitted.

Throughout this paper, all spaces are assumed to be Hausdorff and all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers, Let X be a space and $P \subset X$. Let $\mathcal{P}$ be a family of subsets of X and let $x \in X$. $\bigcup \mathcal{P}$ and $(\mathcal{P})_x$ denote the union $\bigcup \{P : P \in \mathcal{P}\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$ respectively. For a sequence $\{P_n : n \in \mathbb{N}\}$ of subsets of a space X, we abbreviate $\{P_n : n \in \mathbb{N}\}$ to $\{P_n\}$. A point $b = (\beta_n)_{n \in \mathbb{N}}$ of a Tychonoff-product space is abbreviated to (β_n).

2. Main results

Definition 2.1. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$. $\mathcal{P}$ is called a network of X [**7**], if for every $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$, such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.

Definition 2.2. Let $\mathcal{P}$ be a network of a space X. Assume that there exists a countable $\mathcal{P}_x \subset \mathcal{P}$ such that $\mathcal{P}_x$ is a network at x in X for every $x \in X$. Put $\mathcal{P} = \{P_\beta : \beta \in \Lambda\}$. For every $n \in \mathbb{N}$, put $\Lambda_n = \Lambda$ and endow Λ_n a discrete topology. Put $M = \{b = (\beta_n) \in \Pi_{n \in \mathbb{N}} \Lambda_n : \{P_{\beta_n}\}$ *is a network at some point* x_b *in* $X\}$, then M, which is a subspace of the product space $\Pi_{n \in \mathbb{N}} \Lambda_n$, is a metric space and x_b is unique for every $b \in M$. Define $f : M \longrightarrow X$ by $f(b) = x_b$, then f is a mapping, and $(f, M, X, \mathcal{P})$ is called a *Ponomarev*-system [**3, 5, 9**].

Definition 2.3. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a compact mapping (resp. s-mapping), if $f^{-1}(y)$ is a compact (resp. separable) subset of X for every $y \in Y$.

(2) f is called a compact-covering mapping [**7**], if for every compact subset K in Y, there exists a compact subset C in X such that $f(C) = K$.

Definition 2.4. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) Let K be a subset of X. $\mathcal{P}$ is called a cfp-cover of K [**4**], if $\mathcal{P}$ is a cover of K in X such that it can be precisely refined by some finite cover of K consisting of closed subsets of K.

(2) Let K be a subset of X. $\mathcal{P}$ is called to have property cc for K [**5**], if whenever H is a compact subset of K, and $H \subset U$ with U open in X, there exists a subfamily $\mathcal{P}_H$ of $\mathcal{P}$ such that $\mathcal{P}_H$ is a cfp-cover of H and $\bigcup \mathcal{P}_H \subset U$.

Definition 2.5. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a cfp-network of X [**10**], if whenever K is a compact subset of X and $K \subset U$ with U open in X, there exists a subfamily $\mathcal{P}_K$ of $\mathcal{P}$ such that $\mathcal{P}_K$ is a cfp-cover of K and $\bigcup \mathcal{P}_K \subset U$.

(2) $\mathcal{P}$ is called a strong k-network of X [**6**], if whenever K is a compact subset of X, there exists a countable subfamily $\mathcal{P}_K$ of $\mathcal{P}$ such that $\mathcal{P}_K$ has property cc for K.

Remark 2.1. The implication "Strong k-network $\Longrightarrow$ cfp-network" from Definition 2.5. and it can not be reversed from Example 3.1.

Lemma 2.1. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system and let $U = (\Pi_{n\in\mathbb{N}}\Gamma_n)\bigcap M$, where $\Gamma_n \subset \Lambda_n$ for every $n \in \mathbb{N}$. Then $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$ for every $k \in \mathbb{N}$.*

Proof. Let $b = (\beta_n) \in U$ and let $k \in \mathbb{N}$. Then $\{P_{\beta_n}\}$ is a network at $f(b)$ in X and $\beta_k \in \Gamma_k$. So $f(b) \in P_{\beta_k} \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$. This proves that $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$. □

Proposition 2.1. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then f is a compact mapping (resp. s-mapping) iff $\mathcal{P}$ is a point-finite (resp. point-countable) network of X.*

Proof. By Theorem 1.1, we only need to prove necessities.

We only give a proof for the parenthetic part. If $\mathcal{P}$ is not point-countable, then for some $x \in X$ there exists an uncountable subset Γ of Λ, such that $\Gamma = \{\beta \in \Lambda : x \in P_\beta\}$. Let $\{P_{\beta_n}\}$ is a network at x in X. For every $\beta \in \Gamma$, put $c_\beta = (\gamma_n)$, where $\gamma_1 = \beta$, and $\gamma_n = \beta_{n-1}$ for $n > 1$, then $\{P_{\gamma_n}\}$ is a network at x in X, so $c_\beta \in f^{-1}(x)$. Put $U_\beta = (\{\beta\} \times (\Pi_{n>1}\Lambda_n))\bigcap M$ for every $\beta \in \Gamma$, then $\{U_\beta : \beta \in \Gamma\}$ covers $f^{-1}(x)$. If not, there exists $c = (\alpha_n) \in f^{-1}(x)$ and $c \notin U_\beta$ for every $\beta \in \Gamma$, so $\alpha_1 \notin \Gamma$. Thus, $x \notin P_{\alpha_1}$ from the construction of Γ. But $x = f(c) \in P_{\alpha_1}$ from Lemma 2.1. This is a contradiction. Thus, $\{U_\beta : \beta \in \Gamma\}$ is an uncountable open cover of $f^{-1}(x)$, but it has not any proper subcover. So $f^{-1}(x)$ is not separable, hence f is not an s-mapping. □

Now we investigate Question 1.2(3).

Lemma 2.2. *Let X be a product space $\Pi_{n\in\mathbb{N}}\Gamma_n$, where Γ_n is a discrete space for every $n \in \mathbb{N}$. If K is a compact subset of X, then $p_n(K)$ is finite for every $n \in \mathbb{N}$, where $p_n : X \longrightarrow \Gamma_n$ is a project.*

Proof. Let $n \in \mathbb{N}$. Since $p_n : X \longrightarrow \Gamma_n$ is continuous, $p_n(K)$ is compact in Γ_n. Note that Γ_n is a discrete space, $p_n(K)$ is finite. □

The following Lemma is due to [**3**, Lemma 3.4.5]

Lemma 2.3. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system. If K be a compact subset of X such that some countable subset $\mathcal{P}_K$ of $\mathcal{P}$ has property cc for K, then there exists a compact subset C of M, such that $f(C) = K$.*

Theorem 2.1. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then f is a compact-covering mapping iff $\mathcal{P}$ is a strong k-network of X.*

Proof. Sufficiency follows from Lemma 2.3. We only need to prove the necessary conditions.

Let f be a compact-covering mapping. Whenever K is a compact subset of X, there exists a compact subset C of M, such that $f(C) = K$. Put $\Gamma = \bigcup\{p_n(C) : n \in \mathbb{N}\}$, where every $p_n : \Pi_{n\in\mathbb{N}}\Lambda_n \longrightarrow \Lambda_n$ is a project, then $p_n(C)$ is finite for every $n \in \mathbb{N}$ from Lemma 2.2, and so Γ is countable. Put $\mathcal{P}_K = \{P_\beta : \beta \in \Gamma\}$, then $\mathcal{P}_K \subset \mathcal{P}$ is countable. We only need to prove that $\mathcal{P}_K$ possesses property *cc* for K. Let H be a compact subset of K and $H \subset U$ with U open in X. Put $L = f^{-1}(H)\bigcap C$. Let $b = (\beta_n^b)_{n\in\mathbb{N}} \in L$, then $\beta_n^b \in \Gamma$ for every $n \in \mathbb{N}$, and $\{P_{\beta_n^b} : n \in \mathbb{N}\}$ is a network at some point $x_b = f(b) \in H$ in X, so there exists $n_b \in \mathbb{N}$, such that $x_b \in P_{\beta_{n_b}^b} \subset U$. Put

$U_b = ((\Pi_{n<n_b}\Lambda_n) \times \{\beta^b_{n_b}\} \times (\Pi_{n>n_b}\Lambda_n)) \bigcap M$, then U_b is an open neighborhood of b, and $x_b \in f(U_b) \subset P_{\beta^b_{n_b}} \subset U$ from Lemma 2.1. Thus we construct U_b for every $b \in L$. For every $b \in L$, $U_b \bigcap L$ is an open (in subspace L) neighborhood of b, so there exists an open (in subspace L) neighborhood V_b of b, such that $b \in V_b \subset Cl_L(V_b) \subset U_b \bigcap L$, where $Cl_L(V_b)$ is the closure of V_b in subspace L. Since $\{V_b : b \in L\}$ is an open cover of subspace L and L is compact in M, there exists a finite subset $\mathbb{F}$ of L, such that $\bigcup\{V_b : b \in \mathbb{F}\} = L$. Put $\mathcal{P}_H = \{P_{\beta^b_{n_b}} : b \in \mathbb{F}\}$, then $\mathcal{P}_H$ is a finite subfamily of $\mathcal{P}_K$ and $\bigcup \mathcal{P}_H = \bigcup\{P_{\beta^b_{i_b}} : b \in \mathbb{F}\} \subset U$. It suffices to prove that $\mathcal{P}_H$ is a cfp-cover of H. It is clear that $\bigcup\{f(Cl_L(V_b)) : b \in \mathbb{F}\} = f(\bigcup\{Cl_L(V_b) : b \in \mathbb{F}\}) = f(L) = H$. For every $b \in \mathbb{F}$, put $H_b = f(Cl_L(V_b))$. Since $Cl_L(V_b)$ is compact in L, H_b is compact in K, so H_b is closed in K, and $H = \bigcup\{H_b : b \in \mathbb{F}\}$. For every $b \in \mathbb{F}$, $H_b = f(Cl_L(V_b)) \subset f(U_b \bigcap K) \subset f(U_b) \subset P_{\beta^b_{n_b}}$, Note that $P_{\beta^b_{n_b}} \in \mathcal{P}_H$ for every $b \in \mathbb{F}$. This proves that $\mathcal{P}_H$ is a cfp-cover of H. □

3. Other results

Proposition 3.1. *Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system. Then the following are equivalent.*

(1) f is a compact-covering, s-mapping.

(2) $\mathcal{P}$ is a point-countable strong k-network of X.

(3) $\mathcal{P}$ is a point-countable cfp-network of X.

Proof. (1) $\Longrightarrow$ (2) from Proposition 2.1 and Theorem 2.1.

(2) $\Longrightarrow$ (3) from Remark 2.1.

(3) $\Longrightarrow$ (1) from Theorem 1.1 □

Remark 3.1. (1) Proposition 3.1 and Proposition 2.1 give a affirmative answer for Question 1.2(1).

(2) "s-" and "point-countable" in Proposition 3.1 can be replaced by "compact" and "point-finite" respectively.

Corollary 3.1. *Let $\mathcal{P}$ be a point-countable cover of a space X. Then $\mathcal{P}$ is a strong k-network of X iff it is a cfp-network of X.*

Now we shows that the implication in Remark 2.1 can not be reversed by an example. So the answer of Question 1.2(2) is negative from Theorem 2.1.

Lin and Yan proved that any base of a space is a cfp-network of this space [**5**, Lemma 6]. By a similar way, it is easy to prove the following lemma.

Lemma 3.1. *Let $\mathcal{P}$ is a cover of X, which contains a base of X. Then $\mathcal{P}$ is a cfp-network.*

Recall the definition of sequential fan S_ω [**1**]. Let $T_0 = \{a_n : n \in \mathbb{N}\}$ be a sequence converging to $x_0 \notin T_0$, and let T_n be a sequence converging to $a_n \notin T_n$ for every $n \in \mathbb{N}$. Let T be the topological sum of $\{T_n \bigcup\{a_n\} : n \in \mathbb{N}\}$. S_ω is defined as a quotient space obtained from T by identifying all point $a_n \in T$ to the point x_0.

Example 3.1. There exists a Ponomarev-system $(f, M, X, \mathcal{P})$ such that $\mathcal{P}$ is a cfp-network, but $\mathcal{P}$ is not a strong k-network, and so f is not compact-covering.

PROOF. Let X be the sequential fan space S_ω, then X has not any countable neighborhood base at x_0 [**3**], where x_0 is the non-isolated point in X. Put $\mathcal{P} = \{U \subset X : U \textit{ is open in } X\} \bigcup \{\{x_0\}\}$, then $\mathcal{P}$ is a network of X, and there exists a countable $\mathcal{P}_x \subset \mathcal{P}$ such that $\mathcal{P}_x$ is a network at x in X for every $x \in X$. Thus $(f, M, X, \mathcal{P})$ is a Ponomarev-system.

CLAIM 1. $\mathcal{P}$ is a cfp-network of X.

It is obtained from Lemma 3.1.

CLAIM 2. $\mathcal{P}$ is not a strong k-network of X.

Let S be a non-trivial sequence converging to x_0 in X. Put $K = S \bigcup \{x_0\}$, then K is a compact subset of X. If $\mathcal{P}$ is a strong k-network of X, then there exists a countable $\mathcal{P}_K \subset \mathcal{P}$ such that $\mathcal{P}_K$ has property *cc*. It is easy to see that every element in $\mathcal{P}_K - \{\{x_0\}\}$ is open in X and $\mathcal{P}_K - \{\{x_0\}\}$ is a countable network at x_0 in X. So $\mathcal{P}_K$ is a countable neighborhood base at x_0 in X. This contradicts that X has not any countable neighborhood base at x_0. So $\mathcal{P}$ is not a strong k-network of X.

Thus we complete the proof of this example. □

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