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A question on images of metric spaces

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Abstract. In this brief note, we prove that a space is a sequentially-quotient image of a metric space iff it is a pseudo-sequence-covering image of a metric space, which answers a question in [2].

In [**5**], S. Lin proved the following theorem [**5**, Theorem 3.5.14] (see [**2**, Corollary 3.3], for example).

Theorem 1. *A space is a pseudo-sequence-covering, s-image of a metric space iff it is a sequentially-quotient, s-image of a metric space.*

Take this result into account, Y. Ge [**2**] raised the following question recently.

Question 2. Can "s-" in Theorem 1 be omitted? That is, are pseudo-sequence-covering images of metric spaces and sequentially-quotient images of metric spaces equivalent?

In this brief note, we answer the above question. Throughout this paper, all spaces are assumed to be Hausdorff and all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers, $\{x_n\}$ denotes a sequence, where the n-th term is x_n. For a sequence $L = \{x_n\}$, $f(L)$ denotes the sequence $\{f(x_n)\}$. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$. Let $\mathcal{P}$ be a family of subsets of X and let $x \in X$. $\bigcup \mathcal{P}$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$ respectively. For a sequence $\{P_n : n \in \mathbb{N}\}$ of subsets of a space X, we abbreviate $\{P_n : n \in \mathbb{N}\}$ to $\{P_n\}$. A point $b = (\beta_n)_{n\in\mathbb{N}}$ of a Tychonoff-product space is abbreviated to (β_n).

Definition 3. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a pseudo-sequence-covering mapping [**4**], if for every convergent sequence S converging to y in Y, there exists a compact subset K of X such that $f(K) = S \bigcup \{y\}$.

(2) f is called a sequentially-quotient mapping [**1**], if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

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REMARK 4. (1) "Pseudo-sequence-covering mapping" in Definition 3(1) was also called "sequence-covering mapping" by G.Gruenhage, E.Michael and Y.Tanaka in [**3**].

(2) Every pseudo-sequence-covering mapping defined a metric space is a sequentially-quotient mapping [**2**, remark 2.4(3)].

DEFINITION 5. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$. $\mathcal{P}$ is called a network of X [**6**], if for every $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.

THEOREM 6. *Let X be a space. Then there exist a metric space M and a Pseudo-sequence-covering mapping $f : M \longrightarrow X$.*

PROOF. For every $x \in X$ and every sequence $S = \{x_n\}$ converging to x, put $P_{S,k} = \{x_n : n > k\} \bigcup \{x\}$ for every $k \in \mathbb{N}$ and $\mathcal{P}_S = \{P_{S,k} : k \in \mathbb{N}\}$. Put $\mathcal{P}_x = \bigcup\{P_S : S$ is a sequence converging to $x\}$ and $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$. It is clear that $\{x\} \in \mathcal{P}$ for every $x \in X$. We construct a metric space as follows. Let $\mathcal{P} = \{P_\beta : \beta \in \Lambda\}$. For every $n \in \mathbb{N}$, put $\Lambda_n = \Lambda$ and endow Λ_n a discrete topology. Put $M = \{b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Lambda_n : \{P_{\beta_n}\}$ is a network at some point x_b in $X\}$. It suffices to prove the following four facts.

Fact 1. M is a metric space:

In fact, Λ_n, as a discrete space, is a metric space for every $n \in \mathbb{N}$. So M, which is a subspace of the Tychonoff-product space $\Pi_{n\in\mathbb{N}}\Lambda_n$, is a metric space.

Fact 2. Let $b = (\beta_n) \in M$. Then there exists unique x_b such that $\{P_{\beta_n}\}$ is a network at x_b in X:

The existence comes from the construction of M, we only need to prove the uniqueness. Let $\{P_{\beta_n}\}$ be a network at both x_b and x'_b in X, then $\{x_b, x'_b\} \subset P_{\beta_n}$ for every $n \in \mathbb{N}$. If $x_b \neq x'_b$, then there exists an open neighborhood U of x_b such that $x'_b \notin U$. Because $\{P_{\beta_n}\}$ is a network at x_b in X, there exists $n \in \mathbb{N}$ such that $x_b \in P_{\beta_n} \subset U$, thus $x'_b \notin P_{\beta_n}$, a contradiction. This proves the uniqueness.

Define $f : M \longrightarrow X$ as follows. By Fact 2, for every $b = (\beta_n) \in M$, there exists unique x_b such that $\{P_{\beta_n}\}$ is a network at x_b in X. Put $f(b) = x_b$.

Fact 3. f is continuous and onto, so f is a mapping:

For every $x \in X$, $\{x\} \in \mathcal{P} = \{P_\beta : \beta \in \Lambda\}$, so there exists $\beta_n \in \Lambda_n\}$ such that $\{x\} = P_{\beta_n}$. Thus $\{P_{\beta_n}\}$ is a network at x in X. Put $b = (\beta_n)$, then $b \in M$ and $f(b) = x$. So f is onto. Let $b = (\beta_n) \in M$, $f(b) = x_b$. If U is an open neighborhood of x, then there exists $k \in \mathbb{N}$ such that $x_b \in P_{\beta_k} \subset U$ because $\{P_{\beta_n}\}$ is a network at x_b in X. Put $V = \{c = (\gamma_n) \in M : \gamma_k = \beta_k\}$, then U is an open neighborhood of b. It is easy to see that $f(V) \subset P_{\beta_k} \subset U$. So f is continuous.

Fact 4. f is pseudo-sequence-covering:

Whenever $S = \{x_k\}$ is a sequence converging to x in X, put $L = \{x_k : k \in \mathbb{N}\} \bigcup \{x\}$. Without loss of generality, assume $x_k \neq x_l$ for all $k \neq l$. Put $\mathcal{P}_n = \{P_{S,n}\} \bigcup \{\{x_i\} : i = 1, 2, \cdots, n\}$ for every $n \in \mathbb{N}$. Then every $\mathcal{P}_n$ is a finite subfamily of $\mathcal{P}$ and covers L. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\beta : \beta \in \Gamma_n\}$, then Γ_n is a finite subset of Λ_n. Put $K = \{(\beta_n) \in \Pi_{n\in\mathbb{N}}\Gamma_n : L \bigcap (\bigcap_{n\in\mathbb{N}} P_{\beta_n}) \neq \emptyset\}$.

(1) $K \subset M$ and $f(K) \subset L$: Let $b = (\beta_n) \in K$, then $L \bigcap (\bigcap_{n\in\mathbb{N}} P_{\beta_n}) \neq \emptyset$. Pick $y \in L \bigcap (\bigcap_{n\in\mathbb{N}} P_{\beta_n})$, then $y \in L$ and $P_{\beta_n} \in (\mathcal{P}_n)_y$ for every $n \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ such that $y = x_{n_0}$. For every $n > n_0$, $y = x_{n_0} \notin P_{S,n}$, so $(\mathcal{P}_n)_y = \{y\}$,

hence $P_{\beta_n} = \{y\}$. Consequently, $\{P_{\beta_n}\}$ is a network at y in X. Thus $b \in M$ and $f(b) = y \in L$. This proves That $K \subset M$ and $f(K) \subset L$.

(2) $L \subset f(K)$: Let $y \in L$. Pick $\beta_n \in \Gamma_n$ for every $n \in \mathbb{N}$ such that $y \in P_{\beta_n}$. By a similar way as in the proof of (1), we can prove that $\{P_{\beta_n}\}$ is a network at y in X. Put $b = (\beta_n)$, then $b \in K$ and $f(b) = y$. This proves That $L \subset f(K)$.

(3) K is a compact subset of M: Since $K \subset \Pi_{n\in\mathbb{N}}\Gamma_n$ and $\Pi_{n\in\mathbb{N}}\Gamma_n$ is a compact subset of $\Pi_{n\in\mathbb{N}}\Lambda_n$, we only need to prove that K is a closed subset of $\Pi_{n\in\mathbb{N}}\Gamma_n$. Let $b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Gamma_n - K$. Then $L\bigcap(\bigcap_{n\in\mathbb{N}} P_{\beta_n}) = \emptyset$, that is, $\bigcap_{n\in\mathbb{N}}(L\bigcap P_{\beta_n}) = \emptyset$. By the construction, $L\bigcap P_{\beta_n}$ is closed in compact subset L for every $n \in \mathbb{N}$. So there exists $n_0 \in \mathbb{N}$ such that $\bigcap_{n\leqslant n_0}(L\bigcap P_{\beta_n}) = \emptyset$, that is, $L\bigcap(\bigcap_{n\leqslant n_0} P_{\beta_n}) = \emptyset$. Put $W = (\Pi_{n\leqslant n_0}\{\beta_n\}) \times (\Pi_{n>n_0}\Gamma_n)$. Then W is open in $\Pi_{n\in\mathbb{N}}\Gamma_n$ and $b \in W$. It is easy to see that $W\bigcap K = \emptyset$. So K is a closed subset of $\Pi_{n\in\mathbb{N}}\Gamma_n$.

By the above (1), (2) and (3), f is pseudo-sequence-covering. □

Remark 7. By Remark 4(2), every pseudo-sequence-covering image of a metric space is a sequentially-quotient images of a metric space. Combining Theorem 6, a space is a sequentially-quotient image of a metric space iff it is a pseudo-sequence-covering image of a metric space. So the answer of Question 2 is affirmative.

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