SCIENTIA
Series A: *Mathematical Sciences*, Vol. 12 (2006), 17–20
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Torsion-Complete Primary Components in Modular Abelian Group Rings over Certain Rings

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ABSTRACT. Suppose that G is an abelian p-separable group and R is a commutative unital ring of prime characteristic p. It is proved that if R^{p^i} possesses nilpotent elements for each natural number i, then the normalized Sylow p-subgroup $S(RG)$ in the group ring RG is torsion-complete if and only if G is p-bounded. This supplies our recent results from Acta Math. Hung. (1997), Ric. Mat. (2002), Tamkang J. Math. (2003) and Intern. J. Math. and Anal. (2006).

1. Introduction.

Let R be an arbitrary commutative ring with identity of prime characteristic p and G an arbitrary abelian multiplicative group. We denote by RG the group ring of G over R with a normed p-primary component of torsion $S(RG)$ which is also called Sylow p-subgroup. As usual, $N(R)$ denotes the nil-radical of R, G_p the p-component of G and $R^{p^i} = \{r^{p^i} : r \in R, i \in \mathbb{N} \cup \{0\}\}$ as well as $G^{p^i} = \{g^{p^i} : g \in G, i \in \mathbb{N} \cup \{0\}\}$ designate the p^i-th powers of R and G respectively; henceforth $R^{p^\omega} = \cap_{i<\omega} R^{p^i}$ and $G^{p^\omega} = \cap_{i<\omega} G^{p^i}$.

A problem of interest is to find a criterion, only in terms of G and R, for $S(RG)$ to be either torsion-complete or quasi-complete, also termed closed and quasi-closed, respectively.

The investigation in this way starts in [1]-[2] and later on in [9]. Some more detailed achievements were obtained in [3],[4],[5] and [6] under various limitations on both R and G. Although the full solution seems to be in the distant future, we shall settle here the question for torsion completeness via the requirement for an existence of nilpotents in the p^i-th powers of R over every positive integer i. Our result extends the corresponding ones in [3]-[6]. The proof is again based on utilization of the topological version of the criterion for torsion-completeness in terms of bounded Cauchy sequences due to Kulikov (e.g. [8] or [7], vol. II, p. 38, Theorem 70.7). We thus construct a special bounded sequence of Cauchy by using the specific ring properties.

2000 *Mathematics Subject Classification.* Primary 16U60, 16S34; Secondary 20K10, 20K21.
Key words and phrases. torsion-complete groups, Cauchy sequences, group rings, units, nilpotents, bounded groups.

We emphasize that the case when $N(R^{p^\omega}) \neq 0$ was exhausted in [5] and [6]. In the present paper, we examine the general situation when $N(R^{p^i}) \neq 0$, $\forall i \geqslant 1$, whereas $N(R^{p^\omega}) = 0$ and $G^{p^\omega} = 1$. However, the concrete exhibition of such a ring, by giving its explicit representation, is the theme of some other research study.

2. The main result and its proof.

Before proceed by proving our central statement, we recollect for the sake of completeness and for the convenience of the readers our major instruments (see, for instance, [8] and [7], vol. I).

Criterion (L. Ya. Kulikov, 1945). *Suppose A is a multiplicatively written separable abelian p-group. Then A is torsion-complete precisely when every bounded Cauchy sequence of A is convergent in A with respect to the p-adic topology of A, that is if $(a_n)_{n=1}^{\infty} \in A$ so that there exists a natural number m with the property $a_n^{p^m} = 1$ and so that for each positive integer k there is a natural number $\nu = \nu(k)$ with the property $a_{n+l} \in a_n A^{p^k}$ whenever $n \geqslant \nu$ and l is an arbitrary positive integer then there exists $a \in A$ such that $a \in a_n A^{p^k}$.*

We recall that the abelian group G is said to be *p-separable* provided that $G^{p^\omega} = 1$. When G is p-torsion and p-separable, it is simple called *separable*.

And so, we have at our disposal all the information and have accumulated all the machinery needed for verifying the following.

Theorem. *Let G be a p-separable abelian group and R a commutative ring with identity of prime characteristic p such that R^{p^i} has nilpotent elements for all non-negative integers i. Then $S(RG)$ is torsion-complete if and only if G is a bounded p-group.*

Proof. The left implication is straightforward.

In order to check the corresponding right one, we foremost observe that the inclusion $\coprod_{q \neq p} G_q \subseteq G^{p^\omega} = 1$ ensures $\coprod_{q \neq p} G_q = 1$, i.e. G is p-mixed.

Next, we presume in a way of contradiction that G contains an element $g \neq 1$ of infinite order. First of all, if there is $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} : 0 \neq N(R^{p^j}) = N(R^{p^{j+1}})$ then $N(R^{p^j}) = N(R^{p^\omega}) \neq 0$ and everything is done in [5] and [6]. So, we can consider the case when $N(R^{p^i}) \neq N(R^{p^{i+1}}) \neq 0$, $\forall i \in \mathbb{N}_0$, while $N(R^{p^\omega}) = 0$. Thereby, we may choose a family of different nilpotents in R^{p^i} for various $i \geqslant 1$, all of them with order p, namely:

$$r_1^p \neq r_2^{p^2} \neq \cdots \neq r_i^{p^i} \neq \cdots$$

; thus we can select $r_i^{p^i} \in N(R^{p^i}) \setminus N(R^{p^{i+1}})$.

With this in hand, we construct the following infinite sequence

$$\varphi_n = \prod_{i=1}^{n} (1 + r_i^{p^i}(1 - g^{p^i})),$$

where $r_i \in R$ with $r_i^{p^{i+1}} = 0$ and $r_i^{p^i} \neq 0$ as well as $g \in G$ with $g^m \neq 1$, $\forall m \geqslant 1$.

Clearly, $\varphi_n^p = \prod_{i=1}^{n} (1 + r_i^{p^{i+1}}(1 - g^{p^{i+1}})) = 1$, so φ_n is bounded at p.

Moreover, for each $k \geqslant 1$, $n \geqslant k$ and an arbitrary $l \geqslant 0$, we compute that $\varphi_{n+l}.\varphi_n^{-1} = \prod_{i=1}^{n+l}(1+r_i^{p^i}(1-g^{p^i})).\prod_{i=1}^{n}(1+r_i^{p^i}(1-g^{p^i}))^{-1} = \prod_{i=n+1}^{n+l}(1+r_i^{p^i}(1-g^{p^i})) = \prod_{i=n+1}^{n+l}(1+r_i(1-g))^{p^i} \in S^{p^n}(RG) \subseteq S^{p^k}(RG)$.

So, taking into account these two derivations and because of the fact that $S(RG)$ is torsion-complete, the aforementioned Kulikov's criterion is applicable to obtain that there exists a fixed element $f_1b_1 + \cdots + f_tb_t \in S(RG)$, where $t \in \mathbb{N}$, such that

$$f_1b_1 + \cdots + f_tb_t = \prod_{i=1}^{n}(1+r_i^{p^i}(1-g^{p^i})).(\alpha_{1n}^{(k)^{p^k}} a_{1n}^{(k)^{p^k}} + \cdots + \alpha_{s_nn}^{(k)^{p^k}} a_{s_nn}^{(k)^{p^k}})$$

whenever $k \geqslant 1$ and $n \geqslant k$; note that the first index $s_n \in \mathbb{N}$ depends on n.

Now, we intend to write the product $\prod_{i=1}^{n}(1+r_i^{p^i}(1-g^{p^i}))$ in a more convenient for us form. In fact, by induction, we see that $\varphi_1 = 1 + r_1^p(1-g^p)$, $\varphi_2 = (1+r_1^p(1-g^p))(1+r_2^{p^2}(1-g^{p^2})) = (1+r_1^p)(1+r_2^{p^2}) - (1+r_2^{p^2})r_1^pg^p - (1+r_1^p)r_2^{p^2}g^{p^2} + r_1^pr_2^{p^2}g^{p+p^2}$ etc., for every $n \geqslant 3$, $\varphi_n = (1+r_1^p)\cdots(1+r_n^{p^n}) - (1+r_2^{p^2})\cdots(1+r_n^{p^n})r_1^pg^p - \cdots - (1+r_1^p)\cdots(1+r_{n-1}^{p^{n-1}})r_n^{p^n}g^{p^n} + \gamma_{12}g^{p+p^2} + \cdots + \gamma_{1n}g^{p+p^n} + \gamma_{23}g^{p^2+p^3} + \cdots + \gamma_{2n}g^{p^2+p^n} + \cdots - \gamma_{123}g^{p+p^2+p^3} - \cdots - \gamma_{12n}g^{p+p^2+p^n} - \cdots + \gamma_{12\cdots n}g^{p+p^2+\cdots+p^n}$, where the coefficients $\gamma_{12}, \cdots, \gamma_{1n}; \gamma_{23}, \cdots, \gamma_{2n}; \gamma_{123}, \cdots, \gamma_{12n}; \cdots, \gamma_{12\cdots n} = (-1)^n r_1^pr_2^{p^2}\cdots r_n^{p^n}$ depend on $r_1^p, \cdots, r_n^{p^n}$.

Certainly, this record of φ_n is canonical since all degrees of g are different. Besides, the ring coefficients $(1+r_2^{p^2})\cdots(1+r_n^{p^n})r_1^p, \cdots, (1+r_1^p)\cdots(1+r_{n-1}^{p^{n-1}})r_n^{p^n}$ are non-zero because otherwise $(1+r_2^{p^2})\cdots(1+r_n^{p^n})r_1^p = 0$ gives that $r_1^p = 0$ which is untrue; similarly for the other coefficients.

Hereafter, we will choose $n = k > t$. Therefore, the foregoing stated main equality takes the form ($\forall k \geqslant 1$):

$$f_1b_1 + \cdots + f_tb_t = (\beta_0 - \beta_1g^p - \cdots - \beta_kg^{p^k} + \sum_{m=(p,p^2,\cdots,p^k)} \gamma_mg^m).$$

$$(\alpha_{1k}^{(k)^{p^k}} a_{1k}^{(k)^{p^k}} + \cdots + \alpha_{s_kk}^{(k)^{p^k}} a_{s_kk}^{(k)^{p^k}}),$$

where $\beta_0 = (1+r_1^p)\cdots(1+r_k^{p^k})$, $\beta_1 = r_1^p$, $\beta_k = (1+r_1^p)\cdots(1+r_{k-1}^{p^{k-1}})r_k^{p^k}$, $\forall k \geqslant 2$ and all γ_m are from R such that $\beta_0 + \cdots + \beta_k + \sum_m \gamma_m = 1$.

Furthermore, we observe that, since $\alpha_{1k}^{(k)} + \cdots + \alpha_{s_kk}^{(k)} = 1$, for each $i = 1, \cdots, k$ there is an index, say $d = d(i)$, so that $1 \leqslant d \leqslant s_k$ and $\beta_i\alpha_{dk}^{(k)} \neq 0$. Otherwise, if not, $0 = \beta_i\alpha_{1k}^{(k)} + \cdots + \beta_i\alpha_{s_kk}^{(k)} = \beta_i(\alpha_{1k}^{(k)} + \cdots + \alpha_{s_kk}^{(k)}) = \beta_i$ which is false.

Likewise, all products between the degrees $g^{(p,\cdots,p^k)}$ of g, consisting of the sums of all combinations between $p, \cdots, p^k$, and the elements $a_{vk}^{(k)^{p^k}}$ over all indices $1 \leqslant v \leqslant s_k$, so in particular the products $g^{p^e}a_{vk}^{(k)^{p^k}}$ running all superscripts $1 \leqslant e \leqslant t+1$, are parawise different for some positive integer k, whence for almost all natural numbers $\geqslant k$. This is so because in the remaining variant will exist a natural number u with the property that $g^u \in G^{p^k}$ for any $k \geqslant 1$, whence $g^u \in G^{p^\omega} = 1$, that is wrong owing to our choice of the infinite order of g. It is plainly seen that this $k \in \mathbb{N}$ may be chosen to be strictly more than t.

Consequently, bearing in mind the preceding two observations, in the support of the right hand-side there is a number of elements greater than t, which is impossible. This contradiction allows us to conclude that G does not contain elements of infinite order or, in other words, G have to be torsion. That is why, by what we have already shown above that G is p-mixed, we infer that G must be a p-group. Henceforth, our result in [5] (see [6] or [9] too) works.

Finally, G is p-primary bounded and this finishes the proof. $\diamondsuit$

For the full resolution of the problem for torsion-completeness in commutative group algebras, a question of central interest is the following.

Question. Assume that G is an abelian group and R is a commutative unitary ring of prime characteristic p so that R^{p^i} is with nilpotent elements for any $i \geqslant 1$. Does it follow that $S(RG)$ being torsion-complete implies G is p-separable?

We conjecture that this holds in the affirmative. Thus, if the answer of the foregoing query is really positive, by following the method for proof in the Theorem, the problem will be completely solved.

Corrigenda: In [6] there are a few misprints. The most flawed of them is on p.99, line 3, namely the second symbol "b" should be replaced by the right bracket ")".

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Received 7 09 2005, revised 15 03 2006 and 28 04 2006.

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