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Numerical Method for a transport equation perturbed by dispersive terms of 3rd and 5th order

Mauricio Sepúlveda[A] and Octavio Paulo Vera Villagrán[B]

Abstract. We are concerned with the initial-boundary-value problem associated to the Korteweg - de Vries - Kawahara (KdVK) equation, which is a transport equation perturbed by dispersive terms of 3rd and 5th order. The (KdVK) equation appears in several fluid dynamics problems. We obtain local smoothing effects that are uniform with respect to the size of the interval. We also propose a simple finite-difference scheme for the problem and prove its stability. Finally, we give some numerical examples.

1. Introduction

We study the following equation of Korteweg-de Vries-Kawahara type $(KdVK)$ in a bounded subdomain of $\mathbb{R}^2$

$$\left\{\begin{array}{l} u_t + \eta\, u_{xxxxx} + u_{xxx} + uu_x + u_x = 0, \quad x \in [0,\, L], \quad t \in [0,\, T], \\ u(0,\, t) = g_1(t), \quad u_x(0,\, t) = g_2(t), \quad t \in [0,\, T[, \\ u(L,\, t) = 0, \quad u_x(L,\, t) = 0, \quad u_{xx}(L,\, t) = 0, \quad t \in [0,\, T[, \\ u(x,\, 0) = u_0(x), \end{array}\right. \tag{1.1}$$

where $L > 0,\ T \in]0,\, \infty[$ and $\eta \in \mathbb{R}$ is a constant. The above equation is a particular case of the Benney-Lin equation derived by Benney [**1**] and later by Lin [**8, 9**].

$$\left\{\begin{array}{l} u_t + \eta\, u_{xxxxx} + \beta\,(u_{xxxx} + u_{xx}) + u_{xxx} + uu_x + u_x = 0, \quad x \in [0,\, L], \\ u(0,\, t) = g_1(t), \quad u_x(0,\, t) = g_2(t), \qquad t \in [0,\, T[, \\ u(L,\, t) = 0, \quad u_x(L,\, t) = 0, \quad u_{xx}(L,\, t) = 0, \qquad t \in [0,\, T[, \\ u(x,\, 0) = u_0(x) \end{array}\right. \tag{1.2}$$

where $L > 0,\ T \in]0,\, \infty[$ and $\eta \in \mathbb{R}$ is a constant ($\eta < 0$ and $\beta > 0$). It describes the evolution of small but finite amplitude long waves in various problems in fluid dynamics. This also can be seen as a hybrid of the well known fifth order Korteweg-de Vries(KdV) equation or Kawahara equation. For comprehensive descriptions of results pertaining to (1.1), the reader may consult the review articles [**2, 10**] and references

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therein.

2. Results of Existence and Uniqueness

We introduce the spaces $\mathbb{E}$ and $\mathbb{H}$ defined by

$$\mathbb{E} := \{f \in L^1(0,\, T; L^2((1+x^2)dx)),\ \sqrt{t}\, f \in L^2(0,\, T; L^2((1+x^2)dx))\},$$

$$||f||_{\mathbb{E}} := ||f||_{L^1(0,\, T; L^2((1+x^2)dx)} + ||\sqrt{t}\, f||_{L^2(0,\, T; L^2((1+x^2)dx)},$$

where $||u||_{L^2((1+x^2)dx)} = \sqrt{\int_0^L u(x)^2\ (1+x^2)\, dx}$. Let $T > 0$. We define

$$\begin{aligned} \mathbb{H}_T \quad & := \{u \in C([0,\, T];\, L^2((1+x^2)dx)),\ u_x \in C([0,\, T];\, L^2((1+x)dx)),\\ & \sqrt{t}\, u_x \in L^2([0,\, T];\, L^2((1+x)dx)),\\ & \sqrt{t}\, u_{xx} \in L^2([0,\, T];\, L^2(0,\, L)),\ \sqrt{t}\, u_{xxx} \in L^2([0,\, T];\, L^2(0,\, L))\}, \end{aligned}$$

$$\begin{aligned} ||u||_{\mathbb{H}} \quad := \quad & ||u||_{L^\infty(0,\, T; L^2((1+x^2)dx))} + ||u_x||_{L^2(0,\, T; L^2((1+x)dx))}\\ & + ||\sqrt{t}\, u_x||_{L^\infty(0,\, T; L^2((1+x)dx))} + ||\sqrt{t}\, u_{xx}||_{L^2(0,\, T; L^2(0,\, L))}\\ & + ||\sqrt{t}\, u_{xxx}||_{L^2(0,\, T; L^2(0,\, L))}. \end{aligned}$$

Following the proofs in [**7**] it is possible to find similar (or better) estimates for the equations (1.1) and (1.2) obtaining the following result for both equations

Theorem 2.1. *(Existence and Uniqueness) Let $\eta \leqslant 0$, $u_0 \in L^2((1+x^2)dx)$, $g \in H^1_{loc}(\mathbb{R}^+)$ and $0 < L < +\infty$. Then there exists a unique weak maximal solution of (KdVK) defined over $[0,\, T_L]$. Moreover, there exists $T_{min} > 0$ independent of L, depending only on $||u_0||_{L^2(0,\, L)}$ and $||g||_{H^1(0,\, T)}$ such that $T_L \geqslant T_{min}$. The solution u depends continuously on u_0 and g in the following sense: Let a sequence $u_0^n \to u_0$ in $L^2((1+x^2)dx)$, let a sequence $g^n \to g$ in $H^1_{loc}(\mathbb{R}^+)$ and denote by u^n the solution with data $(u_0^n,\, g^n)$ and T_L^n its existence time. Then*

$$\liminf_{n\to+\infty} T_L^n \geqslant T_L$$

and for all $t < T_L$, u^n exists on $[0,\, T]$ if n is large enough and $u^n \to u$ in $\mathbb{H}_T$.

Remark 2.1. *We consider the (KdVK) equation in a quarter plane*

$$\begin{cases} u_t + \eta\, u_{xxxxx} + u_{xxx} + uu_x + u_x = 0, \quad x \geqslant 0, \quad t \geqslant 0,\\ u(0,\, t) = g(t), \qquad t \geqslant 0\\ u(x,\, 0) = u_0(x), \qquad x \geqslant 0. \end{cases}$$

In this context, we are interested in the following result: Consider a family of initial values $u_0^L \in L^2([0,\, L])$ such that $\sup_{\, L\to\infty} \int_0^L |u_0^L(x)|^2\, (1+x^2)\, dx < \infty$ and such that $u_0^L \to u_0$ in $L^2_{loc}(\mathbb{R}^+)$ strongly. Then, for all $T > 0$, if L is large enough, u^L the solution of (KdVK) with initial data u_0^L is defined on $[0,\, T]$ and $u^L \to u$ in $L^p(0,\, T;\, L^2_{loc}(\mathbb{R}^+))$ strongly for all $1 \leqslant p < +\infty$, where u is a solution of (KdVK)$_{QP}$ with initial value u_0.

In order to prove this question, we need similar results for Korteweg-de Vries-Kawahara equation as those obtained by J. Bona and R. Winther [**3, 4, 5**] *for the Korteweg-de Vries equation.*

In the next section we present numerical results for regular solutions of the KdV-Kawahara equation (1.1) describing a numerical scheme for the more general case (1.2).

3. Numerical Methods

We consider finite differences based on the unconditionally stable schemes described in [**6, 7**].

Description of the scheme. We note by v_i^n the approximate value of $u(i\Delta x, n\Delta t)$, solution of the nonlinear problem (BL), where Δx is the space-step, and Δt is the time-step, for $i = 0, \dots, N$, and $n = 0, \dots, M$. Define the discrete space

$$X_N = \{u = (u_0, u_1, \dots, u_N) \in \mathbb{R}^{N+1} \mid u_0 = u_1 = 0 \text{ and } u_N = u_{N-1} = u_{N-2} = 0\}$$

and $(D^+u)_i = \dfrac{u_{i+1} - u_i}{\Delta x}$ and $(D^-u)_i = \dfrac{u_i - u_{i-1}}{\Delta x}$ the classical difference operators. In order to obtain a positive matrix we have to chose a particular discretization. The numerical scheme for the nonlinear problem (1.2) reads as follows :

$$\frac{v^{n+1} - v^n}{\Delta t} + Av^{n+1} + \frac{\alpha}{2} D^-[v^n]^2 = 0, \tag{3.1}$$

where $A = \eta D^+D^+D^+D^-D^- + \beta\,(D^-D^+D^+D^- + D^+D^-) + D^+D^+D^- + \frac{1}{2}(D^+ + D^-)$, with $\alpha = 1$ for the nonlinear case, and $\alpha = 0$ for the linear case. We consider the linear operators D^+ and D^- as matrices of size $(N+1) \times (N+1)$ and we note the following internal product $(z, w) = \sum_{1=0}^{N} z_i w_i$ and $(z, w)_x = (z, xw) = \sum_{1=0}^{N} i\Delta x z_i w_i$, and the norms in $\mathbb{R}^{N+1}$: $|z| = \sqrt{(z, z)}$ and $|z|_x = \sqrt{(z, z)_x}$. Then, we have the following lemma :

LEMMA 3.1. *For all $z, w \in \mathbb{R}^{N+1}$, we have*

$$(D^+z, w) = z_N w_N - z_0 w_0 - (z, D^-w), \tag{3.2}$$

$$(D^+z, z) = \frac{1}{2}\left(\frac{z_N^2}{\Delta x} - \frac{z_0^2}{\Delta x} - \Delta x |D^+z|^2\right), \tag{3.3}$$

$$(D^+z, w)_x = N z_N w_N - (z, D^-w)_x + \Delta x (z, D^-w) - (z, w), \tag{3.4}$$

$$(D^+z, z)_x = \frac{1}{2}\left(\frac{N z_N^2}{\Delta x} - \Delta x |D^+z|_x^2 - |z|^2\right). \tag{3.5}$$

PROOF. Equations (3.2) and (3.4) are result of summing by parts. Equation (3.3) is result of using $(a - b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a - b)^2$. with $z_i = a$ and $z_{i+1} = b$, and summing over $i = 0, \dots, N$. The last equality (3.5) is result of the same identity with $z_i = a$ and $z_{i+1} = b$, multiplying by $i\Delta x$ and summing over $i = 0, \dots, N$. □

In order to obtain estimates for the solution of the numerical scheme for the linear case, we have the following lemmas describing the quadratic forms associated to the different matrices.

LEMMA 3.2. *For all $u \in X_N$, we have*

$$\frac{1}{2}((D^+ + D^-)u, u) = 0, \tag{3.6}$$

$$(D^+D^-u, u) \geqslant -\frac{1}{2}\left(|D^+D^-u|^2 + |u|^2\right), \tag{3.7}$$

$$(D^+D^+D^-u, u) = \frac{\Delta x}{2}|D^+D^-u|^2, \tag{3.8}$$

$$(D^-D^+D^+D^-u, u) = |D^+D^-u|^2, \tag{3.9}$$

$$(D^+D^+D^+D^-D^-u, u) = -\frac{1}{2\Delta x}\left[D^-D^-u\right]_0^2 - \frac{\Delta x}{2}|D^+D^-D^-u|^2. \tag{3.10}$$

REMARK 3.1. *Since $u \in X_N$, the first term in the right-hand side of (3.10) is given by $\frac{1}{\Delta x}\left[D^-D^-u\right]_0^2 = (\Delta x)u_2^2$.*

PROOF. The matrix $\frac{1}{2}(D^+ + D^-)$ is clearly antisymmetric and we have (3.6). The inequality (3.7) is a consequence of $2ab \leqslant a^2 + b^2$. Using (3.3) with $z = D^-u$ we obtain (3.8), and using the same identity with $z = D^-D^-u$ we obtain (3.10). Finally, (3.9) results of summing by parts. □

COROLLARY 3.1. *If $\eta \leqslant 0$, $\beta \geqslant 0$ and Δt is enough small, then $I + \Delta t A$ is positive definite, and for any $u^n \in X_N$ there exists a unique solution u^{n+1} of (3.1).*

PROOF. From Lemma 3.2 we have for all $u \in X_N$ with $u \neq 0$,

$$\begin{aligned} ((I + \Delta t A)u, u) \geqslant &\left(1 - \frac{\beta \Delta t}{2}\right)|u|^2 + \frac{\Delta t\,(\beta + \Delta x)}{2}|D^+D^-u|^2 \\ &- \frac{\eta \Delta x \Delta t}{2}|D^+D^-D^-u|^2 - \frac{\eta \Delta t}{2\Delta x}\left[D^-D^-u\right]_0^2 > 0, \end{aligned} \tag{3.11}$$

when $\beta \Delta t < 2$ and $\eta \leqslant 0$. □

The following estimate shows that the numerical scheme (3.1) with $\alpha = 0$ is l^2-stable and unconditionally stable.

PROPOSITION 3.1. *Let $\eta \leqslant 0$ and $\beta \geqslant 0$. For any $v^n \in X_N$ satisfying the linear scheme (3.1) with $\alpha = 0$, there exists $C(T) > 0$ such that $|v^n| \leqslant C(T)|v^0|$. Moreover, if $\beta > 0$ (η can be zero) we have*

$$\left(\sum_{k=1}^{n} \Delta t |D^+D^-v^k|^2\right)^{\frac{1}{2}} \leqslant C(T)|v^0|.$$

PROOF. Multiplying the numerical scheme (3.1) by v^{n+1} we obtain

$$|v^{n+1}|^2 + \Delta t(Av^{n+1}, v^{n+1}) = (v^{n+1}, v^n), \tag{3.12}$$

and then, using the same identity of the proof of the Lemma **3.1** with $a = v^{k+1}$ and $b = v^k$ and summing for $k = 0, \ldots, n-1$ we have

$$|v^n|^2 + \sum_{k=0}^{n-1}|v^{k+1} - v^k|^2 + 2\Delta t \sum_{k=1}^{n}(Av^k, v^k) = |v^0|^2.$$

From (3.11) this last equality becomes

$$\begin{aligned}
|v^n|^2 &+ \Delta t \sum_{k=0}^{n-1} \Delta t \left| \frac{v^{k+1} - v^k}{\Delta t} \right|^2 + (\beta + \Delta x) \sum_{k=1}^{n} \Delta t |D^+ D^- v^k|^2 \\
&- \eta \Delta x \sum_{k=1}^{n} \Delta t |D^+ D^- D^- v^k|^2 - \frac{\eta}{\Delta x} \sum_{k=1}^{n} \Delta t \left[D^- D^- v^k \right]_0^2 \\
&\leqslant |v^0|^2 + \beta \sum_{k=1}^{n} \Delta t |v^k|^2. \qquad (3.13)
\end{aligned}$$

On the other hand, from (3.11) and (3.12) we deduce for $k = 0, \ldots, N-1$

$$|v^k| \leqslant \left(1 - \frac{\beta \Delta t}{2}\right)^{-k} |v^0| \leqslant e^{2T/\beta} |v^0|,$$

with $T = n\Delta t$. Replacing this inequality in (3.13) we deduce

$$\begin{aligned}
|v^n|^2 &+ \Delta t \sum_{k=0}^{n-1} \Delta t \left| \frac{v^{k+1} - v^k}{\Delta t} \right|^2 + (\beta + \Delta x) \sum_{k=1}^{n} \Delta t |D^+ D^- v^k|^2 \\
&- \eta \Delta x \sum_{k=1}^{n} \Delta t |D^+ D^- D^- v^k|^2 - \frac{\eta}{\Delta x} \sum_{k=1}^{n} \Delta t \left[D^- D^- v^k \right]_0^2 \\
&\leqslant \left(1 + \beta T e^{4T/\beta}\right) |v^0|^2.
\end{aligned}$$

□

In order to obtain the unconditional stability for the nonlinear version of the scheme, we will find a discrete estimate that is equivalent to that of (3.28) (see Proposition **3.1**). Let us denote by x the sequence $x_i = i\Delta x$. We have :

LEMMA 3.3. *For all $u \in X_N$, we have*

$$\begin{aligned}
\frac{1}{2}((D^+ + D^-)u, xu) &= \frac{1}{4}\Delta x^2 |D^+ u|^2 - \frac{1}{2}|u|^2, \\
(D^+ D^- u, xu) &= -|D^- u|_x^2 - \frac{\Delta x}{2}|D^+ u|^2, \\
(D^+ D^+ D^- u, xu) &= \frac{3}{2}|D^- u|^2 + \frac{\Delta x}{2}|D^+ D^- u|_x^2 - \frac{\Delta x^2}{2}|D^+ D^- u|^2, \\
(D^- D^+ D^+ D^- u, xu) &= |D^+ D^- u|_x^2, \\
(D^+ D^+ D^+ D^- D^- u, xu) &= -\frac{5}{2}|D^- D^- u|^2 - \frac{\Delta x}{2}|D^+ D^- D^- u|_x^2 \\
&+ \Delta x^2 |D^+ D^- D^- u|^2 - \frac{1}{2}\left[D^- D^- u\right]_0^2,
\end{aligned}$$

where $(xu)_i = i\Delta x u_i$.

Proof. Using (3.2), (3.3) and (3.4) we have

$$\begin{aligned} ((D^+ + D^-)u, xu) &= (D^-u,u)_x - (u, D^-u)_x + \Delta x(u, D^-u) - |u|^2 \\ &= -\Delta x(D^+u,u) - |u|^2 = \frac{\Delta x^2}{2}|D^+u|^2 - |u|^2, \end{aligned}$$

and then we have the first identity of the Lemma. Following the same idea and applying the identities of Lemma **3.1** is easy to prove the rest of the identities. □

Proposition 3.2. *Let $\eta \leqslant 0$ and $\beta \geqslant 0$. For any $v^n \in X_N$ satisfying the linear scheme (3.1) with $\alpha = 0$, there exists $C(T) > 0$ such that $|v^n|_x \leqslant C(T)|v^0|_x$ and*

$$\begin{aligned} \left(\sum_{k=1}^{n} \Delta t|D^-v^k|^2\right)^{\frac{1}{2}} &\leqslant C(T)|v^0|, \\ \left(\sum_{k=1}^{n} \Delta t|D^+D^-v^k|^2\right)^{\frac{1}{2}} &\leqslant C(T)|v^0|, \qquad \textit{if } \eta < 0 \ (\beta \textit{ can be zero}), \\ \left(\sum_{k=1}^{n} \Delta t|D^+D^-v^k|_x^2\right)^{\frac{1}{2}} &\leqslant C(T)|v^0|, \qquad \textit{if } \beta > 0 \ (\eta \textit{ can be zero}). \end{aligned}$$

Proof. We multiply the numerical scheme (3.1) with $\alpha = 0$ by xv^{n+1}. Then, applying Lemma **3.3**, and the same identity of the proof of Lemma **3.1** with $a = \sqrt{x}v^{k+1}$ and $b = \sqrt{x}v^k$, we deduce

$$\begin{aligned} |v^n|_x^2 &+ \Delta t\sum_{k=0}^{n-1}\Delta t\left|\frac{v^{k+1}-v^k}{\Delta t}\right|_x^2 + \left(3 + \frac{\Delta x^2}{2}\right)\sum_{k=1}^{n}\Delta t|D^-v^k|^2 \\ &+ (2\beta + \Delta x)\sum_{k=1}^{n}\Delta t|D^+D^-v^k|_x^2 - \left(5\eta + \Delta x^2\right)\sum_{k=1}^{n}\Delta t|D^+D^-v^k|^2 \\ &- \eta\Delta x\sum_{k=1}^{n}\Delta t|D^+D^-D^-v^k|_x^2 + 2\eta\Delta x^2\sum_{k=1}^{n}\Delta t|D^+D^-D^-v^k|^2 \\ &- \eta\left[D^-D^-v^k\right]_0^2 = |v^0|_x^2 + \sum_{k=1}^{n}\Delta t\left(|v^k|^2 + 2\beta|D^-v^k|^2\right). \end{aligned}$$

Noting that $|D^+D^-v^k|_x^2 - \Delta x|D^+D^-v^k|^2 = \sum_{i=1}^{N-1}\frac{(i-1)}{\Delta x}(v_{i+1}^k - 2v_i^k + v_{i-1}^k)^2 \geqslant 0$, and $|D^+D^-D^-v^k|_x^2 - 2\Delta x|D^+D^-v^k|^2 + \frac{1}{\Delta x}\left[D^-D^-v^k\right]_0^2 = \sum_{i=2}^{N-1}\frac{(i-2)}{\Delta x}(v_{i+1}^k - 3v_i^k + 3v_{i-1}^k - v_{i-2}^k)^2 \geqslant 0$, and replacing this inequality, we deduce

$$\begin{aligned} |v^n|_x^2 &+ \Delta t\sum_{k=0}^{n-1}\Delta t\left|\frac{v^{k+1}-v^k}{\Delta t}\right|_x^2 + 3\sum_{k=1}^{n}\Delta t|D^-v^k|^2 + \beta\sum_{k=1}^{n}\Delta t|D^+D^-v^k|_x^2 \\ &- 5\eta\sum_{k=1}^{n}\Delta t|D^+D^-v^k|^2 \leqslant |v^0|_x^2 + \sum_{k=1}^{n}\Delta t\left(1+\beta\right)|v^k|^2. \end{aligned}$$

Finally, using the inequalities of Proposition **3.1** and the fact that we have in a boundary domain $(0, L)$, we may conclude the proof. □

Now, let us introduce the non-homogeneous linear scheme approximating the solution of the $(\text{KdVK})_{NH}$ problem :

$$\frac{v^{n+1}-v^n}{\Delta t} + Av^{n+1} = f^n.$$

The existence proof of the continuous case studied in the previous sections applies in the discrete non-homogeneous linear case and the discrete nonlinear case for any discretization of the non-linear part, in particular for $f_n = \frac{1}{2}D^-[u^n]^2$. Thus, we obtain the following result of convergence :

THEOREM 3.1. *For any $u^n \in X_N$ satisfying the non-linear scheme (3.1), with $\alpha = 1$, and $\eta \leqslant 0$, there exists $\varepsilon_0 > 0$ such that, if $\Delta t \leqslant \varepsilon_0$, then there exists $T > 0$ and a constant $C = C(T) > 0$ (independent of Δt and Δx) such that :*

$$\sup_{k=0,\ldots,p} |v^k|^2 + \Delta t \sum_{k=0}^{p} |D^- v^k|^2 - \eta \Delta t \sum_{k=0}^{p} |D^- D^- v^k|^2 \leqslant C|v_0|^2.$$

This result means that the scheme is unconditionally stable. Let us observe that in agreement with the gain of regularity of the (KdVK) equation, we obtain an additional estimate respect to the analogous numerical scheme of the KdV equation, studied in detail in [**7**].

4. Some numerical results

We study the re-normalized Korteweg-de Vries-Kawahara equation with initial and boundary condition on $[0, L]$, with $L = 10$:

$$u_t + \frac{1}{L^5}\eta u_{xxxxx} + \frac{1}{L^3}\eta u_{xxx} + \frac{1}{L} u u_x + \frac{1}{L} u_x = 0.$$

We have taken $\eta = -1$, $\Delta t = 2.5 \times 10^{-5}$, $\Delta x = 5 \times 10^{-4}$. We compute the solution during 9600 iterations in time, that is on the time interval $[0, T]$ with $T = 0.24$. The initial value is

$$u_0(x) = \frac{\alpha}{\cosh^2(\beta L(x - 1/2))} + \frac{4\alpha}{\cosh^2(2\beta L(x - 1/8))}$$

with $\alpha = 12\beta^2$ and $\beta = 2$. This correspond to the superposition of two solitons with different speeds. Using, Matlab we obtain In Figure 1, we have represented the solution at time $t_i = iT/10$ for $i = 0, \ldots, 9$.

The validity of the results can be expressed by the graph of the function $t \mapsto \|u_\Delta(\cdot, t)\|_{L^2(0,L)}$, where u_Δ is the discrete solution of the numerical scheme (see Figure 2). In theory, if the support of the solution stays in the interval $(0, L)$, the L^2 norm must be conserved. It is not rigorously the case of our simulations, but looking at the graphs of Figures 2, we can say that it is approximately certain for the KdV equation $(\eta = 0)$: we have 0.06% of lost of the norm L^2 value. In the case of the simulation of the KdV-K equation (Figure 1), we note that the first pic touch the boundary $x = 0$ and then, we have 0.25% of lost of the norm L^2 value.

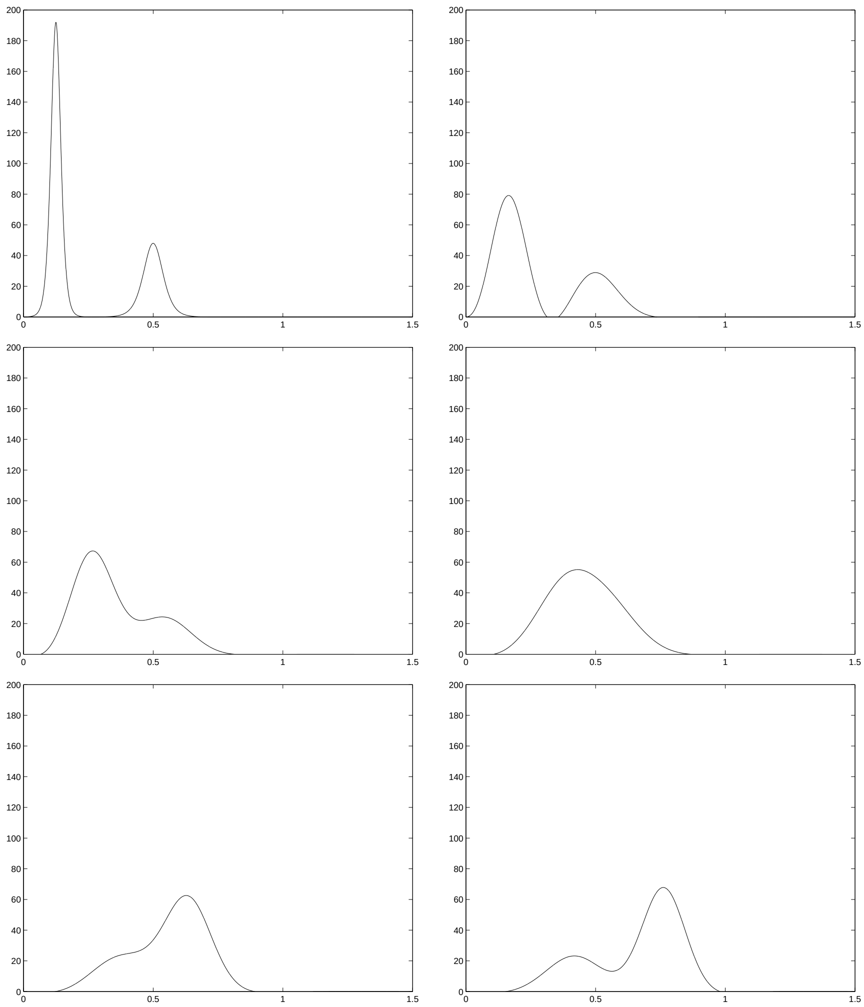

FIGURE 1. Interaction of two solitons for the Korteweg-de Vries-Kawahara equation ($\eta = -1$).

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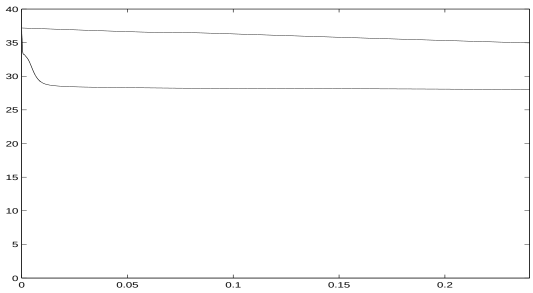

FIGURE 2. $t \mapsto \|u_\Delta(\cdot, t)\|_{L^2(0,L)}$ for $\eta = 0$ and $\eta = -1$.

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[A]DEPARTAMENTO DE INGENIERÍA MATEMÁTICA,
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS,
UNIVERSIDAD DE CONCEPCIÓN,
CASILLA 160-C,
CONCEPCIÓN, CHILE.
E-mail address: mauricio@ing-mat.udec.cl

[B]DEPARTAMENTO DE MATEMÁTICA,
UNIVERSIDAD DEL BÍO-BÍO,
COLLAO 1202, CASILLA 5-C,
CONCEPCIÓN, CHILE
E-mail address: overa@ubiobio.cl,octavipaulov@yahoo.com