

SCIENTIA

Series A: *Mathematical Sciences*, Vol. 14 (2007), 7–15

Universidad Técnica Federico Santa María

Valparaíso, Chile

ISSN 0716-8446

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## The integrals in Gradshteyn and Ryzhik. Part 2: Elementary logarithmic integrals

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ABSTRACT. We describe methods to evaluate elementary logarithmic integrals. The integrand is the product of a rational function and a linear polynomial in  $\ln x$ .

### 1. Introduction

The table of integrals by I. M. Gradshteyn and I. M. Ryzhik [3] contains a large selection of definite integrals of the form

$$(1.1) \quad \int_a^b R(x) \ln^m x \, dx,$$

where  $R(x)$  is a rational function,  $a, b \in \mathbb{R}^+$  and  $m \in \mathbb{N}$ . We call integrals of the form (1.1) *elementary logarithmic integrals*. The goal of this note is to present methods to evaluate them. We may assume that  $a = 0$  using

$$(1.2) \quad \int_a^b R(x) \ln^m x \, dx = \int_0^b R(x) \ln^m x \, dx - \int_0^a R(x) \ln^m x \, dx.$$

Section 2 describes the situation when  $R$  is a polynomial. Section 3 presents the case in which the rational function has a single simple pole. Finally section 4 considers the case of multiple poles.

### 2. Polynomials examples

The first example considered here is

$$(2.1) \quad I(P; b, m) := \int_0^b P(x) \ln^m x \, dx,$$

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2000 *Mathematics Subject Classification*. Primary 33.

*Key words and phrases*. Integrals.

The author wishes to thank Luis Medina for a careful reading of the manuscript. The partial support of NSF-DMS 0409968 is also acknowledged.

where  $P$  is a polynomial. This can be evaluated in elementary terms. Indeed,  $I(P; b, m)$  is a linear combination of

$$(2.2) \quad \int_0^b x^j \ln^m x \, dx,$$

and the change of variables  $x = bt$  yields

$$(2.3) \quad \int_0^b x^j \ln^m x \, dx = b^{j+1} \sum_{k=0}^m \binom{m}{k} \ln^{m-k} b \int_0^1 t^j \ln^k t \, dt.$$

The last integral evaluates to  $(-1)^k k! / (j+1)^{k+1}$  either an easy induction argument or by the change of variables  $t = e^{-s}$  that gives it as a value of the gamma function.

**Theorem 2.1.** Let  $P(x)$  be a polynomial given by

$$(2.4) \quad P(x) = \sum_{j=0}^p a_j x^j.$$

Then

$$(2.5) \quad I(P; b, m) := \int_0^b P(x) \ln^m x \, dx = \sum_{k=0}^m (-1)^k k! \binom{m}{k} \ln^{m-k} b \sum_{j=0}^p a_j \frac{b^{j+1}}{(j+1)^{k+1}}.$$

This expression shows that  $I(P; b, m)$  is a linear combination of  $b^j \ln^k b$ , with  $1 \leq j \leq 1 + p (= 1 + \deg(P))$  and  $0 \leq k \leq m$ .

### 3. Linear denominators

We now consider the integral

$$(3.1) \quad f(b; r) := \int_0^b \frac{\ln x \, dx}{x + r}$$

for  $b, r > 0$ . This corresponds to the case in which the rational function in (1.1) has a single simple pole.

The change of variables  $x = rt$  produces

$$(3.2) \quad \int_0^b \frac{\ln x \, dx}{x + r} = \ln r \ln(1 + b/r) + \int_0^{b/r} \frac{\ln t \, dt}{1 + t}.$$

Therefore, it suffices to consider the function

$$(3.3) \quad g(b) := \int_0^b \frac{\ln t \, dt}{1 + t},$$

as we have

$$(3.4) \quad f(b; r) = \ln r \ln \left( 1 + \frac{b}{r} \right) + g \left( \frac{b}{r} \right).$$

Before we present a discussion of the function  $g$ , we describe some elementary consequences of (3.2).

**Elementary examples.** The special case  $r = b$  in (3.2) yields

$$(3.5) \quad \int_0^b \frac{dx}{x+b} = \ln 2 \ln b + \frac{\ln t dt}{1+t}.$$

Expanding  $1/(1+t)$  as a geometric series, we obtain

$$(3.6) \quad \frac{\ln t dt}{1+t} = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}.$$

This appears as **4.231.1** in [3]. Differentiating (3.2) with respect to  $r$  produces

$$(3.7) \quad \int_0^b \frac{\ln x dx}{(x+r)^2} = -\frac{\ln(b+r)}{r} + \frac{\ln r}{r} + \frac{b \ln b}{r(r+b)}.$$

As  $b, r \rightarrow 1$  we obtain

$$(3.8) \quad \int_0^1 \frac{\ln x dx}{(1+x)^2} = -\ln 2.$$

This appears as **4.231.6** in [3]. On the other hand, as  $b \rightarrow \infty$  we recover **4.231.5** in [3]:

$$(3.9) \quad \int_0^\infty \frac{\ln x dx}{(x+r)^2} = \frac{\ln r}{r}.$$

**The polylogarithm function.** The evaluation of the integral

$$(3.10) \quad g(b) := \int_0^b \frac{\ln t dt}{1+t},$$

requires the transcendental function

$$(3.11) \quad \text{Li}_n(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^n}.$$

This is the *polylogarithm function* and it has also appeared in [5] in our discussion of the family

$$(3.12) \quad h_n(a) := \int_0^\infty \frac{\ln^n x dx}{(x-1)(x+a)}, \quad n \in \mathbb{R}, a > 0.$$

In the current context we have  $n = 2$  and we are dealing with the *dilogarithm function*:  $\text{Li}_2(x)$ .

**Lemma 3.1.** The function  $g(b)$  is given by

$$(3.13) \quad g(b) = \ln b \ln(1+b) + \text{Li}_2(-b).$$

PROOF. The change of variables  $t = bs$  yields

$$(3.14) \quad g(b) = \ln b \ln(1+b) + \int_0^1 \frac{\ln s ds}{1+bs}.$$

Expanding the integrand in a geometric series yields the final identity.  $\square$

**Theorem 3.2.** Let  $b, r > 0$ . Then

$$(3.15) \quad \int_0^b \frac{\ln x \, dx}{x+r} = \ln b \ln \left( \frac{b+r}{r} \right) + \text{Li}_2 \left( -\frac{b}{r} \right).$$

**Corollary 3.3.** Let  $b > 0$ . Then

$$(3.16) \quad \int_0^b \frac{\ln x \, dx}{x+b} = \ln 2 \ln b - \frac{\pi^2}{12}.$$

PROOF. Let  $r \rightarrow b$  in Theorem 3.2 and use

$$(3.17) \quad \text{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

□

The expression in Theorem 3.2 and the method of partial fractions gives the explicit evaluation of elementary logarithmic integrals where the rational function has simple poles. For example:

**Corollary 3.4.** Let  $0 < a < b$  and  $r_1 \neq r_2 \in \mathbb{R}^+$ . Then, with  $r = r_2 - r_1$ , we have

$$\begin{aligned} \int_a^b \frac{\ln x \, dx}{(x+r_1)(x+r_2)} &= \frac{1}{r} \left[ \ln b \ln \left( \frac{r_2(b+r_1)}{r_1(b+r_2)} \right) + \ln a \ln \left( \frac{r_1(a+r_2)}{r_2(a+r_1)} \right) \right] + \\ &+ \frac{1}{r} \left[ \text{Li}_2 \left( -\frac{b}{r_1} \right) - \text{Li}_2 \left( -\frac{a}{r_1} \right) - \text{Li}_2 \left( -\frac{b}{r_2} \right) + \text{Li}_2 \left( -\frac{a}{r_2} \right) \right]. \end{aligned}$$

The special case  $a = r_1$  and  $b = r_2$  is of interest:

**Corollary 3.5.** Let  $0 < a < b$ . Then

$$\begin{aligned} \int_a^b \frac{\ln x \, dx}{(x+a)(x+b)} &= \frac{1}{b-a} [\ln(ab) \ln(a+b) - \ln 2 \ln(ab) - 2 \ln a \ln b] \\ &+ \frac{1}{b-a} \left[ -2\text{Li}_2(-1) + \text{Li}_2 \left( -\frac{b}{a} \right) + \text{Li}_2 \left( -\frac{a}{b} \right) \right]. \end{aligned}$$

The integral in Corollary 3.5 appears as **4.232.1** in [3]. An interesting problem is to derive **4.232.2**

$$(3.18) \quad \int_0^{\infty} \frac{\ln x \, dx}{(x+u)(x+v)} = \frac{\ln^2 u - \ln^2 v}{2(u-v)}$$

directly from Corollary 3.5.

We now present an elementary evaluation of this integral and obtain from it an identity of Euler. We prove that

$$(3.19) \quad \int_a^b \frac{\ln x \, dx}{(x+a)(x+b)} = \frac{\ln ab}{2(b-a)} \ln \frac{(a+b)^2}{4ab}.$$

PROOF. The partial fraction decomposition

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left( \frac{1}{x+a} - \frac{1}{x+b} \right).$$

reduces the problem to the evaluation of

$$I_1 = \int_a^b \frac{\ln x \, dx}{x+a} \text{ and } I_2 = \int_a^b \frac{\ln x \, dx}{x+b}.$$

The change of variables  $x = at$  gives, with  $c = b/a$ ,

$$\begin{aligned} I_1 &= \int_1^c \frac{\ln(at) \, dt}{1+t} \\ &= \ln a \int_1^c \frac{dt}{1+t} + \int_1^c \frac{\ln t}{1+t} \, dt \\ &= \ln a \ln(1+c) - \ln a \ln 2 + \int_1^c \frac{\ln t}{1+t} \, dt. \end{aligned}$$

Similarly,

$$I_2 = \ln b \ln 2 - \ln b \ln(1+1/c) + \int_1^{1/c} \frac{\ln t}{1+t} \, dt.$$

Therefore

$$\begin{aligned} I_1 - I_2 &= \ln a \ln(1+c) + \ln b \ln(1+1/c) - \ln 2 \ln a - \ln 2 \ln b + \\ &+ \int_1^c \frac{\ln t}{1+t} \, dt - \int_{1/c}^1 \frac{\ln t}{1+t} \, dt. \end{aligned}$$

Let  $s = 1/t$  in the second integral to get

$$\int_{1/c}^1 \frac{\ln t}{1+t} \, dt = \int_c^1 \frac{\ln s}{s(1+s)} \, ds.$$

Replacing in the expression for  $I_1 - I_2$  yields

$$\begin{aligned} I_1 - I_2 &= \ln a (\ln(a+b) - \ln a - \ln 2) - \ln b (\ln 2 - \ln(a+b) + \ln b) + \\ &+ \int_1^c \frac{\ln t}{t} \, dt. \end{aligned}$$

The last integral can now be evaluated by elementary means to produced the result.  $\square$

Now comparing the two evaluation of the integral in Corollary 3.5 produces an identity for the dilogarithm function.

**Corollary 3.6.** The dilogarithm function satisfies

$$(3.20) \quad \text{Li}_2(-z) + \text{Li}_2\left(-\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2} \ln^2(z).$$

This is the first of many interesting functional equations satisfied by the polylogarithm functions. It was established by L. Euler in 1768. The reader will find in [4] a nice description of them.

#### 4. A single multiple pole

In this section we consider the evaluation of

$$(4.1) \quad f_n(b, r) := \int_0^b \frac{\ln x \, dx}{(x+r)^n}.$$

This corresponds to the elementary rational integrals with a single pole (at  $x = -r$ ). The change of variables  $x = rt$  yields

$$f_n(b, r) = \frac{\ln r}{(n-1)r^{n-1}} \left[ \frac{(b+r)^{n-1} - r^{n-1}}{(b+r)^{n-1}} \right] + \frac{1}{r^{n-1}} h_n(b/r),$$

where

$$(4.2) \quad h_n(b) := \int_0^b \frac{\ln t \, dt}{(1+t)^n}.$$

We first establish a recurrence for  $h_n$ .

**Theorem 4.1.** Let  $n > 2$  and  $b > 0$ . Then  $h_n$  satisfies the recurrence

$$(4.3) \quad h_n(b) = \frac{n-2}{n-1} h_{n-1}(b) + \frac{b \ln b}{(n-1)(1+b)^{n-1}} + \frac{1 - (1+b)^{n-2}}{(n-1)(n-2)(1+b)^{n-2}}.$$

PROOF. Start with

$$h_n(b) = \int_0^b \frac{[(1+t) - t] \ln t \, dt}{(1+t)^n} = h_{n-1}(b) - \int_0^b \frac{t \ln t \, dt}{(1+t)^n}.$$

Integrate by parts in the last integral, with  $u = t \ln t$  and  $dv = dt/(1+t)^n$  to produce the result.  $\square$

The initial condition for this recurrence is obtained from the value

$$(4.4) \quad h_2(b) = \frac{b}{1+b} \ln b - \ln(1+b).$$

This expression follows by a direct integration by parts in

$$(4.5) \quad h_2(b) = -\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^b \ln t \frac{d}{dt} (1+t)^{-1} dt.$$

The first few values of  $h_n(b)$  suggest the introduction of the function

$$(4.6) \quad q_n(b) := (1+b)^{n-1} h_n(b),$$

for  $n \geq 2$ . For example,

$$(4.7) \quad q_2(b) = b \ln b - (1+b) \ln(1+b).$$

The recurrence for  $h_n$  yields one for  $q_n$ .

**Corollary 4.2.** The recurrence

$$(4.8) \quad q_n(b) = \frac{(n-2)}{(n-1)} (1+b) q_{n-1}(b) + \frac{b \ln b}{n-1} - \frac{(1+b) [(1+b)^{n-2} - 1]}{(n-1)(n-2)},$$

holds for  $n \geq 2$ .

Corollary 4.2 establishes the existence of functions  $X_n(b)$ ,  $Y_n(b)$  and  $Z_n(b)$ , such that

$$(4.9) \quad q_n(b) = X_n(b) \ln b + Y_n(b) \ln(1+b) + Z_n(b).$$

The recurrence (4.8) produces explicit expression for each of these parts.

**Proposition 4.3.** Let  $n \geq 2$  and  $b > 0$ . Then

$$(4.10) \quad X_n(b) = \frac{(1+b)^{n-1} - 1}{n-1}.$$

PROOF. The function  $X_n$  satisfies the recurrence

$$(4.11) \quad X_n(b) = \frac{n-2}{n-1}(1+b)X_{n-1}(b) + \frac{b}{n-1}.$$

The initial condition is  $X_2(b) = b$ . The result is now easily established by induction.  $\square$

**Proposition 4.4.** Let  $n \geq 2$  and  $b > 0$ . Then

$$(4.12) \quad Y_n(b) = -\frac{(1+b)^{n-1}}{n-1}.$$

PROOF. The function  $Y_n$  satisfies the recurrence

$$(4.13) \quad Y_n(b) = \frac{n-2}{n-1}(1+b)Y_{n-1}(b).$$

This recurrence and the initial condition  $Y_2(b) = -(1+b)$ , yield the result.  $\square$

It remains to identify the function  $Z_n(b)$ . It satisfies the recurrence

$$(4.14) \quad Z_n(b) = \frac{n-2}{n-1}(1+b)Z_{n-1}(b) - \frac{(1+b)[(1+b)^{n-2} - 1]}{(n-2)(n-1)}.$$

This recurrence and the initial condition  $Z_2(b) = 0$  suggest the definition

$$(4.15) \quad T_n(b) := -\frac{(n-1)! Z_n(b)}{b(1+b)}.$$

**Lemma 4.5.** The function  $T_n(b)$  is a polynomial of degree  $n-3$  with positive integer coefficients.

PROOF. The function  $T_n(b)$  satisfies the recurrence

$$(4.16) \quad T_n(b) = (n-2)(1+b)T_{n-1}(b) + (n-3)! \left[ \frac{(1+b)^{n-2} - 1}{b} \right].$$

Now simply observe that the right hand side is a polynomial in  $b$ .  $\square$

Properties of the polynomial  $T_n(b)$  will be described in future publications. We now simply observe that its coefficients are *unimodal*. Recall that a polynomial

$$(4.17) \quad P_n(b) = \sum_{k=0}^n c_k b^k$$

is called *unimodal* if there is an index  $n^*$ , such that  $c_k \leq c_{k+1}$  for  $0 \leq k \leq n^*$  and  $c_k \geq c_{k+1}$  for  $n^* < k \leq n$ . That is, the sequence of coefficients of  $P_n$  has a single peak. Unimodal polynomials appear in many different branches of Mathematics. The reader will find in [2] and [6] information about this property. We now use the result of [1] to establish the unimodality of  $T_n$ .

**Theorem 4.6.** Suppose  $c_k > 0$  is a nondecreasing sequence. Then  $P(x+1)$  is unimodal.

Therefore we consider the polynomial  $S_n(b) := T_n(b-1)$ . It satisfies the recurrence

$$(4.18) \quad S_n(b) = b(n-2)S_{n-1}(b) + (n-3)! \sum_{r=0}^{n-3} b^r.$$

Now write

$$(4.19) \quad S_n(b) = \sum_{k=0}^{n-3} c_{k,n} b^k,$$

and conclude that  $c_{0,n} = (n-3)!$  and

$$(4.20) \quad c_{k,n} = (n-2)c_{k-1,n-1} + (n-3)!,$$

from which it follows that

$$(4.21) \quad c_{k+1,n} - c_{k,n} = (n-2)[c_{k,n-1} - c_{k-1,n-1}].$$

We conclude that  $c_{k,n}$  is a nondecreasing sequence.

**Theorem 4.7.** The polynomial  $T_n(b)$  is unimodal.

**Conclusions.** We have given explicit formulas for integrals of the form

$$(4.22) \quad \int_a^b R(x) \ln x \, dx,$$

where  $R$  is a rational function with real poles. Future reports will describe the case of higher powers

$$(4.23) \quad \int_a^b R(x) \ln^m x \, dx,$$

as well as the case of complex poles, based on integrals of the form

$$(4.24) \quad C_n(a, r) := \int_0^b \frac{\ln x \, dx}{(x^2 + r^2)^n}.$$



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*Received 27 07 2006, revised 06 11 2006*

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