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# ON SOME QUESTIONS OF $wN$-SPACES

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Abstract. In this paper, we show that a quotient space obtained from the butterfly space is a separable, $MCP$-space and is not a $wN$-space, which answers a question on $wN$-spaces negatively. We also prove that a locally countably compact space $X$ is an $MCP$-space if and only if it is a $k_c\beta$-space, which give a partial answer for a question on $MCP$-spaces.

$MCP$-spaces and $wN$-spaces were investigated and the following question were raised by C. Good, R. Knight and I. Stares in [5].

**Question 1.** (1) Is there a separable $MCP$-space that is not $wN$?
(2) Is there a (reasonable) $g$-function characterization of $MCP$ spaces?

In this paper, we answer Question 1(1) negatively and give a partial answer of Question 1(2). Throughout this paper, all spaces are assumed to be regular and all mappings are continuous and onto. The letter $\omega$ denotes the first infinite ordinal, the letter $\mathbb{R}$ denotes the set of all real numbers. Let $A$ be a subset of a space, $\bar{A}$ and $A^\circ$ denote closure and interior of $A$, respectively. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of sets, we write $\{A_n\} \preceq \{B_n\}$ if $A_n \subset B_n$ for every $n \in \omega$.

**Definition 2.** A space $X$ is called to be an $MCP$-space ([5]), if there exists an operator $U$ assigning to every decreasing sequence $\mathcal{D}=\{D_n\}$ of closed subsets of $X$ with empty intersection, a sequence $U(\mathcal{D})=\{U(n,\mathcal{D})\}$ of open subsets of $X$ such that the following hold.
(1) $D_n \subset U(n,\mathcal{D})$ for every $n \in \omega$.
(2) $\bigcap_{n\in\omega} \overline{U(n,\mathcal{D})} = \emptyset$.
(3) If $\mathcal{D} \preceq \mathcal{E}$, then $U(\mathcal{D}) \preceq U(\mathcal{E})$.

**Definition 3.** Let $X$ be a space. $g : \omega \times X \longrightarrow \{\textit{open subsets of } X\}$ is called to be a $g$-function on $X$ if $x \in \bigcap_{n\in\omega} g(n,x)$ for every $x \in X$. A space $X$ is called to be a $wN$-space ([6]) (resp. a $q$-space ([8]), a $k_c\beta$-space ([11])) if there is a $g$-function on $X$ such that the following (1) (resp. (2), (3)) holds.
(1) If $g(n,x) \bigcap g(n,x_n) \neq \emptyset$ for every $n \in \omega$, then the sequence $\{x_n\}$ has a cluster point.
(2) If $x_n \in g(n,x)$ for every $n \in \omega$, then the sequence $\{x_n\}$ has a cluster point.
(3) Let $K$ be a countably compact subset of $X$. If $K \bigcap g(n,x_n) \neq \emptyset$ for every $n \in \omega$, then the sequence $\{x_n\}$ has a cluster point.

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**Definition 4.** A space $X$ is called to be stratifiable ([2]) if there is a function $S : \omega \times \tau \longrightarrow \{\text{closed subsets of } X\}$, where $\tau$ is the topology on $X$, such that following (1) and (2) hold.

(1) If $U \in \tau$, then $U = \cup_{n\in\omega}(S(n, U))^\circ$.

(2) If $U$, $V \in \tau$ and $U \subset V$, then $S(n, U) \subset S(n, V)$ for every $n \in N$.

*Remark* 5. The following are known (see [5], for example).

(1) Stratifiable $\Longrightarrow MCP$.

(2) $wN \Longleftrightarrow q$, $MCP$.

*Construction* 6. Let $X = \{(x, y) : x, y \in \mathbb{R}\}$. $X$ is the butterfly space ([3]), if the topology on $X$ is generated as follows.

(1) If $(x, y) \in X$, where $y \neq 0$, then neighborhoods of $(x, y)$ are the natural neighborhoods of $(x, y)$ on the Euclidean plane $\mathbb{R}^2$.

(2) If $(x, 0) \in X$, then take as a base at the point $(x, 0)$ the family $\{\{(x, 0)\} \cup U_n(x) : n \in \omega\}$, where $U_n(x)$ is the set of all points inside the circle of radius $1/n$ and center at $(x, 0)$ but outside the two circles of radius $n$ tangent to the $x$-axis at $(x, 0)$.

*Remark* 7. The butterfly space is hereditarily separable ([9, Page 175-176]) and stratifiable ([7, Example 4.3]).

By Remark 5, the following example give a negative answer for Question 1(1).

*Example* 8. There is a hereditarily separable, stratifiable space, which is not a $q$-space.

*Proof.* Let $X$ be the butterfly space and let $I$ be a closed bounded interval on the real axis. Let $Y$ be the quotient space obtained from $X$ by shrinking $I$ to a point. Put $f : X \longrightarrow Y$ is the natural quotient mapping. It is clear that $f$ is a closed mapping. Note that closed mappings preserve hereditarily separable spaces and stratifiable spaces ([2]). So $Y$ is a hereditarily separable, stratifiable space from Remark 7, but $Y$ is not a $q$-space from [7, Example 4.3]. □

The following lemma come from [4, Theorem 1].

**Lemma 9.** *A space $X$ is an $MCP$-space if and only if there is an operator $V$ assigning to every closed subset $F$ of $X$, a sequence $V(F)$=$\{V_n(F)\}$ of open subsets of $X$ such that the following hold.*

*(1) $V_{n+1}(F) \subset V_n(F)$ and $F \subset V_n(F)$ for every closed subset $F$ of $X$ and every $n \in \omega$.*

*(2) If $E$ and $F$ are closed subsets of $X$ and $E \subset F$, then $V_n(E) \subset V_n(F)$ for every $n \in \omega$.*

*(3) If $\{F_n\}$ is a decreasing sequence of closed subsets of $X$ and $\bigcap_{n\in\omega} F_n = \emptyset$, then $\bigcap_{n\in\omega} \overline{V_n(F_n)} = \emptyset$.*

**Proposition 10.** *If $X$ is an $MCP$-space, then $X$ is a $k_c\beta$-space.*

*Proof.* Let $X$ be an $MCP$-space. By Lemma 9, there is an operator V assigning to every closed subset $F$ of $X$, a sequence $V(F)$=$\{V_n(F)\}$ of open subsets of $X$ satisfying Lemma 9(1),(2),(3). For every $n \in \omega$ and for every $x \in X$, put $g(n, x) = V_n(\{x\})$. Then $g(n, x)$ is a $g$-function on $X$. Let $K$ be a countably compact subset of $X$ and $K \bigcap g(n, x_n) \neq \emptyset$ for every $n \in \omega$, It suffices to prove that $\{x_n : n \in \omega\}$ has a cluster point. If not, put $F_n = \{x_m : m \geq n\}$. Then $\{F_n : n \in \omega\}$ is a decreasing sequence of closed subsets of $X$ with empty intersection. By Lemma 9(3), $\bigcap_{n\in\omega}\overline{V_n(F_n)}=\emptyset$. On the other hand, for every $n\in\omega$, $K\bigcap g(n,x_n)\neq\emptyset$, that is, $K\bigcap V_n(\{x_n\})\neq\emptyset$. For every $n\in\omega$, notice that $x_n\in F_n$, so $K\bigcap V_n(F_n)\neq\emptyset$ from Lemma 9(2). Thus $\{K\bigcap\overline{V_n(F_n)}:n\in\omega\}$ is a countable collection of closed subsets of countably compact subspace $K$ which has the finite intersection property. Hence $\bigcap_{n\in\omega}(K\bigcap\overline{V_n(F_n)})\neq\emptyset$, so $K\bigcap(\bigcap_{n\in\omega}\overline{V_n(F_n)})\neq\emptyset$. This contradicts that $\bigcap_{n\in\omega}\overline{V_n(F_n)}=\emptyset$. □

**Proposition 11.** *If $X$ is a locally countably compact, $k_c\beta$-space, then $X$ is an $MCP$-space.*

*Proof.* Let $X$ be a locally countably compact, $k_c\beta$-space, and let $g(n,x)$ be a $g$-function on $X$ which satisfies Definition 3(3). For every closed subset $F$ of $X$, put $V(F)=\{V_n(F)\}$, where $V_n(F)=\bigcup\{g(n,x):x\in F\}$ for every $n\in\omega$. It is easy to see that the operator $V$ described as above satisfies Lemma 9(1),(2). We only need to prove that the operator $V$ satisfies Lemma 9(3). Let $\{F_n:n\in\omega\}$ is a decreasing sequence of closed subsets of $X$ with empty intersection. It suffices to prove that $\bigcap_{n\in\omega}\overline{V_n(F_n)}=\emptyset$. If not, $\bigcap_{n\in\omega}\overline{V_n(F_n)}\neq\emptyset$, then there exists $p\in X$ such that $p\in\overline{V_n(F_n)}$ for every $n\in\omega$. Since $X$ is locally countably compact, there exists a countable compact neighborhood $K$ of $p$. Note that $K\bigcap V_n(F_n)\neq\emptyset$ for every $n\in\omega$. So there exists $x_n\in F_n$ such that $K\bigcap g(n,x_n)\neq\emptyset$. Thus sequence $\{x_n:n\in\omega\}$ has a cluster point $x$ from Definition 3(3). It is easy to see that $x\in\bigcap_{n\in\omega}F_n$. This contradicts that $\bigcap_{n\in\omega}F_n=\emptyset$. □

Note that every locally countably compact, $MCP$-space is a $wN$-space ([5, Theorem 8]). So every locally countably compact, $k_c\beta$-space is a $wN$-space.

**Theorem 12.** *A locally countably compact space $X$ is an $MCP$-space (or $wN$-space) if and only if it is a $k_c\beta$-space.*

We do not know whether "locally countably compact" in Proposition 11 can be omitted. That is, we have the following question.

**Question 13.** Is a $k_c\beta$-space an $MCP$-space?

Recall a A mapping $f:X\longrightarrow Y$ is finite-to-one (resp. $k$-to-one), if $f^{-1}(y)$ consists of finite points (exactly $k$ points) in $X$ for each $y\in Y$. Tanaka[10, Example 3.2] gave an example to show that finite-to-one and open mappings do not preserve $wN$-spaces, that is, [5, Proposition 18] is not true. An interesting question is: How open mappings preserve $wN$-spaces? Note that every $k$-to-one and open mappings is a closed mapping ([1, Lemma 1 and Lemma 2]) and finite-to-one and closed mappings preserve $wN$-spaces ([4, Corollary 11]). We have the following proposition.

**Proposition 14.** *Let $f:X\longrightarrow Y$ be a $k$-to-one and open mapping. If $X$ is a $wN$-space, then $Y$ is a $wN$-space.*

## References

1. A.Arhangelskii, Test for the existence of bicompact element in a continuous decomposition, theorem on the invariance of weight in open-closed finitely multiple mappings, Dokl. Akad. Nauk. SSSR, 166(1966), 1263-1266.
2. C.R.Borge, On stratifiable spaces, Pacific J. Math., 17(1966), 1-16.
3. J.G.Ceder, Some generalizations of metric spaces, Pacific J. Math., 11(1961), 105-125.

4. Y.Ge and C.Good, A note on monotone countable paracompactness, Comment Math. Univ. Carolinae, 42(2001), 771-778.
5. C.Good, R.Knight and I.Stares, Monotone countable paracompactness, Topology Appl., 101(2000), 281-298.
6. R.E.Hodel, Moore spaces and $w\triangle$-spaces, Pacific J. Math., 38(1971), 641-652.
7. D.J.Lutzer, Semimetrizable and stratifible spaces, Topology Appl., 1(1971), 43-48.
8. E.Michael, A note on closed maps and compact sets, Israel J. Math., 2(1964), 173-176.
9. L.A.Steen and Jr.J.A.Seebach, Counterexamples in topology, Springer-Verlag, New York, 1978.
10. Y.Tanaka, On open finite-to-one maps, Bull. Tokyo Gakugei Univ., Ser. IV, 25(1973), 1-13.
11. L.Wu, On $k$-semistratifiable spaces, J. of Suzhou Univ., 1(1983), 1-4.(in Chinese)

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