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Characterizations of spaces with locally countable weak bases

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Abstract. In this paper, by 1-sequence-covering mappings, the relations between metric spaces and spaces with (σ-) locally countable weak bases are established. These are answers to Alexandroff's some problems.

1. Introduction

In 1961, Alexandroff [**1**] conceived that the relations among various classes of topological spaces could be established by means of various mappings. A central question of Alexandoff's idea is that the relations between various topological spaces and metric spaces are established by means of various mappings. In the paper [**5**], S. Lin introduced metrizable stratified strong s-mappings(i.e. msss-mappings), by which the relations between metric spaces and spaces with σ-locally countable networks, or spaces with σ-locally countable k-networks, or spaces with σ-locally countable bases were established. In this paper, we will establish the relations between metric spaces and spaces with (σ-) locally countable weak bases. These are answers to Alexandroff's some problems.

In this paper, all spaces are regular T_1, and all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers. p_i denotes the projection from the product space $\prod_{n\in\mathbb{N}} X_n$ to X_i.

2. Spaces with locally countable weak bases

Definition 2.1. [**10**] A collection $\mathcal{B}$ of subsets of a space X is called a weak base for X, if for each $x \in X$, there exists $\mathcal{B}_x \subset \mathcal{B}$ such that : (1) $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$; (2) $x \in \cap\mathcal{B}_x$; (3) if $U, V \in \mathcal{B}_x$, then $U \cap V \in \mathcal{B}_x$; (4) a subset G of X is open if and only if for every $x \in G$ there exists a $B \in \mathcal{B}_x$ such that $B \subset G$. If each $\mathcal{B}_x$ is countable then X is called a g-first countable space. An element of $\mathcal{B}_x$ is called a weak neighborhood of x.

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DEFINITION 2.2. [**11**] A mapping $f : X \to Y$ is called a sequence-covering if every convergent sequence (including its limit) in Y is the image of a convergent sequence in X under f.

DEFINITION 2.3. [**6**] A mapping $f : X \to Y$ is called 1-sequence-covering if for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that if $y_n \to y$ in Y, then there is $x_n \in f^{-1}(y_n)$ with $x_n \to x$.

It is easy to see that a 1-sequence-covering mapping is a sequence-covering mapping.

DEFINITION 2.4. [**5**] A mapping $f : X \to Y$ is called an ss-mapping if for every $y \in Y$, there exists an open neighborhood V of y such that $f^{-1}(V)$ is a separable subspace of X.

DEFINITION 2.5. Let X be a space, and let $\mathcal{P}$ be a cover of X. $\mathcal{P}$ is called a cs-network [**4**] if whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a neighborhood of x, then $\{x\} \cup \{x_n : n \geqslant m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathcal{P}$. $\mathcal{P}$ is called a cs*-network [**3**] if whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a neighborhood of x, then there exists $P \in \mathcal{P}$ such that $x \in P \subset U$ and P contains a subsequence of $\{x_n\}$.

DEFINITION 2.6. [**2**] For a space X, $P \subset X$ is called a sequential neighborhood of x in X if $x_n \to x$ in X, then there exists $k \in \mathbb{N}$ such that $\{x\} \cup \{x_n : n \geqslant k\} \subset P$; P is called a sequential open set of X if P is a sequential neighborhood of any point $x \in P$. X is called a sequential space if each sequential open subset of X is open.

LEMMA 2.7. [**6**] *If B_x is a weak neighborhood of x in a space X, then B_x is a sequential neighborhood of x.*

LEMMA 2.8. [**6**] *Let $f : X \to Y$ be a mapping. If $\{B_n : n \in \mathbb{N}\}$ is a decreasing network of some point x and each $f(B_n)$ is a sequential neighborhood of $f(x)$ in Y, supposed $y_n \to f(x)$, then there exists $x_n \in f^{-1}(y_n)$ such that $x_n \to x$ in X.*

PROOF. For $n \in \mathbb{N}$, since $f(B_n)$ is a sequential neighborhood of $f(x)$, there is $i(n) \in \mathbb{N}$ such that $y_i \in f(B_n)$ whenever $i \geqslant i(n)$. Hence $f^{-1}(y_i) \cap B_n \neq \phi$. We may assume that $1 < i(n) < i(n+1)$. For $j \in \mathbb{N}$, set

$$x_j \in \begin{cases} f^{-1}(y_j), & j < i(1) \\ f^{-1}(y_j) \cap B_n, & i(n) \leqslant j < i(n+1), n \in \mathbb{N} \end{cases}$$

then $x_j \in f^{-1}(y_j)$, and $x_j \to x$ in X. □

THEOREM 2.9. *For a space X, the following are equivalent:*
(1) X has a locally countable weak base;
(2) X is g-first countable and has a locally countable k-network;
(3) X is g-first countable and has a locally countable cs-network;
(4) X is g-first countable and has a locally countable cs-network;*
(5) X is a 1-sequence-covering quotient ss-image of a metric space.

PROOF. (1) $\Longleftrightarrow$ (2) follows the Theorem 2.1 [**8**].

(1) $\Longleftrightarrow$ (3) follows the proof of the Lemma 7 [**7**].

(2) $\Longleftrightarrow$ (4) follows the the Lemma 1 [**9**].

(1)$\Longrightarrow$(5). Let $\mathcal{P} = \{P_\alpha : \alpha \in A\}$ be a locally countable weak base of X. Suppose $\mathcal{P}$ is closed under finite intersections. Let A_n be the set A with discrete topology for each $n \in \mathbb{N}$. Set

$M = \{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\}$ forms a network at some point $x(\alpha)$ in X $\}$, and give M the subspace topology induced from the usual product topology of the discrete spaces A_n, then M is a metric space. We define $f : M \to X$ by $f(\alpha) = x(\alpha)$.

1) f is a continuous and surjective mapping. Since $\mathcal{P}$ is a point countable weak base of X, it is easy to check that f is continuous and surjective.
2) f is an ss-mapping. For each $x \in X$, since $\mathcal{P}$ is locally countable, there exists a neighborhood V of x in X such that $V \cap P \neq \phi$ for only countable many $P \in \mathcal{P}$. So $f^{-1}(V)$ is a separable subspace of M, f is an ss-mapping.
3) f is a 1-sequence-covering mapping. For $x \in X$, let $\{P_{\alpha_n} : n \in \mathbb{N}\} \subset \mathcal{P}$ be the local weak base of x in X, set $\delta = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n$, then $\delta \in f^{-1}(x)$. For $n \in \mathbb{N}$, set

$$B_n = \{(\beta_i) \in M : \beta_i = \alpha_i \;\; when \;\; i \leqslant n\},$$

then $\{B_n : n \in \mathbb{N}\}$ is a local decreasing base of δ in M, and $f(B_n) = \bigcap_{i\leqslant n} P_{\alpha_i}$. In fact, let $\beta = (\beta_i) \in B_n$, then $f(\beta) = \bigcap_{i\in\mathbb{N}} P_{\beta_i} \subset \bigcap_{i\leqslant n} P_{\beta_i}$, so $f(B_n) \subset \bigcap_{i\leqslant n} P_{\alpha_i}$. Let $z \in \bigcap_{i\leqslant n} P_{\alpha_i}$, choosing a subcollection $\{P_{\gamma_i} : i \in \mathbb{N}\}$ of $\mathcal{P}$ such that $\gamma_i = \alpha_i$ when $i \leqslant n$, and $\{P_{\gamma_i} : i \in \mathbb{N}\}$ forms a network of z in X. Set $\gamma = (\gamma_i)$, then $\gamma \in B_n$ and $z = f(\gamma) \in f(B_n)$, so $\bigcap_{i\leqslant n} P_{\alpha_i} \subset f(B_n)$. Since $\{P_{\alpha_n} : n \in \mathbb{N}\} \subset \mathcal{P}$ be the local weak base of x in X, $f(B_n)$ is a sequential neighborhood by the Lemma 2.7. Suppose $x_j \to x$ in X, by the Lemma 2.8, there exist $\delta_j \in f^{-1}(x_j)$ and $\delta_j \to \delta$ in M, therefore f is a 1-sequence-covering mapping. Since X is g-first countable, X is a sequential space. By the Proposition 5 [**11**], f is a quotient mapping.
(5)$\Longrightarrow$(1) Let M be a metric space and $f : M \to X$ be a 1-sequence-covering, quotient, ss-mapping. Suppose that $\mathcal{B}$ be a σ- locally finite base of M. Since f is a ss-mapping, $f(\mathcal{B})$ is locally countable. For $x \in X$, there exists $\alpha_x \in f^{-1}(x)$ satisfying the Definition 2.3.
Set

$$\mathcal{P}_x = \{f(B) : \alpha_x \in B \in \mathcal{B}\}, \;\; \mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}.$$

Suppose that U be an open subset of X and $x \in U$, then there exists $\alpha_x \in f^{-1}(x) \subset f^{-1}(U)$. Since $\mathcal{B}$ is a base of M, there is $B \in \mathcal{B}$ such that $\alpha_x \in B \subset f^{-1}(U)$, therefore

$x \in f(B) \subset U$, $f(B) \in \mathcal{P}_x$. Inversely, Suppose that for each $x \in U$ there exists $f(B) \in \mathcal{P}_x$ such that $f(B) \subset U$. Let $x_n \to x$ in X, then there exist $\alpha_x \in f^{-1}(x)$ and $\alpha_{x_n} \in f^{-1}(x_n)$ such that $\alpha_{x_n} \to \alpha_x$ in M. Since $f(B) \in \mathcal{P}_x$ and $\alpha_x \in B \in \mathcal{B}$, $\{\alpha_{x_n}\}$ is eventually in B. Hence $\{x_n\}$ is eventually in $f(B) \subset U$, U is sequential open. Since f is a quotient mapping, X is sequential. Therefore U is an open subset of X, $\mathcal{P}$ satisfies (4) of the Definition 2.1. It is easy to check that $\mathcal{P}$ satisfies (1), (2) and (3) of the Definition 2.1. Hence $\mathcal{P}$ is a locally countable weak base. □

3. Spaces with σ-locally countable weak bases

Definition 3.1. [**5**] A mapping $f : X \to Y$ is called stratified strong s-mapping, if X is a subspace of a product space $\prod_{i\in\mathbb{N}} X_i$ and for each $y \in Y$, there exists a sequence $< V_i >$ of open neighborhood of y in Y such that each $p_i f^{-1}(V_i)$ is a separable subspace of X_i. If each X_i is a metric space, then f is called a metrizable stratified strong s-mapping, or an msss-mapping for short.

Theorem 3.2. *For a space X, the following are equivalent:*
(1) X has a σ-locally countable weak base;
(2) X is g-first countable and has a σ-locally countable cs-network;
(3) X is a 1-sequence-covering quotient msss-image of a metric space.

Proof. (1) $\Longleftrightarrow$ (2) follows the proof of the Lemma 7 [**7**].
(1)$\Longrightarrow$(3). Let $\mathcal{P} = \bigcup_{i\in\mathbb{N}} \mathcal{P}_i$ be a σ-locally countable weak base of X, and $\mathcal{P}_x = \bigcup_{i\in\mathbb{N}} \mathcal{P}_{ix}$, $\mathcal{P}_{ix} \subset \mathcal{P}_i$. For $i \in \mathbb{N}, x \in X$, we may assume that $\mathcal{P}_{ix} \subset \mathcal{P}_{i+1x}$ and $X \in \mathcal{P}_i$. Set $\mathcal{P}_i = \{P_\alpha : \alpha \in A_i\}$. Set
$M = \{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\}$ forms a network at some point $x(\alpha)$ in X $\}$, and give M the subspace topology induced from the usual product topology of the discrete spaces A_n, then M is a metric space. We define $f : M \to X$ by $f(\alpha) = x(\alpha)$.
1) f is a continuous and surjective mapping. Since $\mathcal{P}$ is a point countable weak base of X, it is easy to check that f is continuous and surjective.
2) f is a msss-mapping. For each $x \in X$, $i \in \mathbb{N}$, since $\mathcal{P}_i$ is locally countable, there exists a neighborhood V_i of x in X such that $|\{\alpha \in A_i : P_\alpha \cap V_i \neq \phi| \leqslant \aleph_0$. Set $B_i = \{\alpha \in A_i : P_\alpha \cap V_i \neq \phi\}$, then $p_i f^{-1}(V_i) \subset B_i$, so $p_i f^{-1}(V_i)$ is a separable subspace of A_i. Hence f is an msss-mapping.
3) f is a 1-sequence-covering mapping. Since X is g-first countable, X is a sequential space. By the Proposition 5 [**11**], if f is a 1-sequence-covering mapping, then f is a quotient mapping. So we need only to prove f is a 1-sequence-covering mapping. For $x \in X$, as each $\mathcal{P}_{ix}$ is countable, set

$$\mathcal{P}_{ix} = \{P_{ij} : j \in \mathbb{N}\}, \;\; i \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, by (3) of the Definition 2.1, there exists $k(n) \in \mathbb{N}$ and $P_n \in \mathcal{P}_{k(n)x}$ such that $P_n = \bigcap_{i,j\leqslant n} P_{ij}$. As $\mathcal{P}_{ix} \subset \mathcal{P}_{i+1x}$, we may assume $k(n) < k(n+1)$, $n \in \mathbb{N}$.

Set

$$P_{\alpha_i} = \begin{cases} P_{ii} & 1 \leqslant i \leqslant k(1) \\ P_n & k(n) \leqslant i < k(n+1),\ n \in \mathbb{N} \end{cases},$$

then $P_{\alpha_i} \in \mathcal{P}_{ix} \subset \mathcal{P}_i$ and $\{P_{\alpha_i} : i \in \mathbb{N}\}$ forms a network of the point x. Set $\gamma = (\alpha_i)$, then $\gamma \in M$ and $\gamma \in f^{-1}(x)$. For each $n \in \mathbb{N}$, set

$$B_n = \{(\beta_i) \in M : \beta_i = \alpha_i \ \ when \ \ i \leqslant n\},$$

then $\{B_n : n \in \mathbb{N}\}$ is a local decreasing base of γ in M, and $f(B_n) = \bigcap_{i \leqslant n} P_{\alpha_i}$. By the Lemma 2.7, $f(B_n)$ is a sequential neighborhood of x. Suppose $x_j \to x$ in X, by the Lemma 2.8, there exist $\gamma_j \in f^{-1}(x_j)$ and $\gamma_j \to \gamma$ in M, therefore f is a 1-sequence-covering mapping.

(3)$\Longrightarrow$(1) Let M be a metric space and $f : M \to X$ be a 1-sequence-covering, quotient, msss-mapping, then there exists a sequence of metric spaces $\{X_i\}$ satisfying the Definition 3.1. For each $i \in \mathbb{N}$, X_i has a σ-locally finite base $\mathcal{P}_i$, set

$$\mathcal{B}_n = \{M \cap (\bigcap_{j \leqslant n} p_j^{-1}(P)) : P \in \mathcal{P}_j, j \leqslant n\},\ \ \mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n,$$

then $\mathcal{B}$ is a base of M. Since f is a 1-sequence-covering mapping, $f(\mathcal{B})$ is a cs-network of X. For $x \in X$, there exists a sequence $< V_i >$ of open neighborhood of x such that $p_i f^{-1}(V_i)$ is a separable subspace of X_i. For each $n \in \mathbb{N}$, set $V = \bigcap_{i \leqslant n} V_i$, then $|\{Q \in f(\mathcal{B}_n) : V \cap Q \neq \phi\}| \leqslant \aleph_0$. Therefore $f(\mathcal{B})$ is a σ-locally countable collections of X. For each $x \in X$, there exists $\alpha_x \in f^{-1}(x)$ satisfying the Definition 2.3. Set

$$\mathcal{P}_x = \{f(B) : \alpha_x \in B \in \mathcal{B}\},\ \ \mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}.$$

Suppose that U be an open subset of X and $x \in U$, then there exists $\alpha_x \in f^{-1}(x) \subset f^{-1}(U)$. Since $\mathcal{B}$ is a base of M, there is a $B \in \mathcal{B}$ such that $\alpha_x \in B \subset f^{-1}(U)$, therefore $x \in f(B) \subset U$, $f(B) \in \mathcal{P}_x$. Inversely, suppose that for each $x \in U$ there exists $f(B) \in \mathcal{P}_x$ such that $f(B) \subset U$. Let $x_n \to x$ in X, then there exist $\alpha_x \in f^{-1}(x)$ and $\alpha_{x_n} \in f^{-1}(x_n)$ such that $\alpha_{x_n} \to \alpha_x$ in M. Since $f(B) \in \mathcal{P}_x$ and $\alpha_x \in B \in \mathcal{B}$, $\{\alpha_{x_n}\}$ is eventually in B. Hence $\{x_n\}$ is eventually in $f(B) \subset U$, U is sequential open. Since f is a quotient mapping, X is sequential. Therefore U is an open subset of X, $\mathcal{P}$ satisfies (4) of the Definition 2.1. It is easy to check that $\mathcal{P}$ satisfies (1), (2) and (3) of the Definition 2.1. Hence $\mathcal{P}$ is a σ-locally countable weak base.

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