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A remarkable Definite Integral

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Abstract. A definite integral is evaluated explicitly whose integrand, subject to mild restrictions, contains an arbitrary function. A number of examples are given.

Introduction

The ultimate goal in the definite integration of functions of a single variable would be a formula such as

$$\int_{-\infty}^{\infty} F(x)dx = G[F] \tag{0}$$

where F denotes any integrable function and $G[F]$ is an explicit expression free of the integration symbol. A trivial, but restricted, example is the defining property of the Dirac delta function $\int f(x)\delta(x-a)dx = f(a)$. One might consider Ramanujan's "Master Theorem" [1], which expresses the integral of a function in terms of the analytic continuation of its Taylor coefficients, the first representative this class. This note deals with a more explicit formula of this nature and some of its consequences.

Calculation

We start with the formula proven in [2]: For $a > 0$, $t > 0$

$$\int_0^{\infty} \frac{e^{-tx^2}\cos(t\pi x)\cosh x}{1+2a^2\cosh(2x)+a^4}dx = \frac{\pi e^{-t[\frac{\pi^2}{4}+(\ln a)^2]}}{4a(1+a^2)}. \tag{1}$$

If one multiplies both sides of (1) by any function $f(t)$, which possesses a Laplace transform

$$F(k) = \int_0^{\infty} e^{-kt} f(t)dt, \tag{2}$$

and integrates with respect to t over $[0,\infty]$, one has

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Theorem 1. If the inverse Laplace transform of F exists and is integrable for positive real argument, then

$$\int_{-\infty}^{\infty} \frac{F(x^2+i\pi x)\cosh x}{1+2a^2\cosh(2x)+a^4}dx = \frac{\pi F[(\pi/2)^2+(\ln a)^2]}{2a(1+a^2)}. \tag{3}$$

Equation (3) is our principal result.

Examples

With $a=1$ and $x \to \pi x$, (3) can be written

$$\int_{-\infty}^{\infty} \frac{F[cx(x+i)]}{\cosh(\pi x)} = \pi F(c/4) \tag{3a}$$

In this form, to avoid complications where $x=0$ is a singularity, it is convenient to require that F be analytic at $x=i$.

For $f(t)=e^{-bt}$, one has $F(k)=1/(b+k)$, $b>0$. Hence,

$$I(a,b) = \int_0^{\infty} \frac{(x^2+b)\cosh x}{(x^4+(2b+\pi^2)x^2+b^2)(1+2a^2\cosh(2x)+a^4)}dx$$
$$= \frac{\pi}{4a(1+a^2)(b+\pi^2/4+\ln^2 a)}. \tag{4}$$

For $f(t)=J_0(t)$, $F(k)=1/\sqrt{1+k^2}$. Hence

$$\int_{-\infty}^{\infty} \frac{\cosh x}{1+2a^2\cosh(2x)+a^4}\frac{dx}{\sqrt{1+(x^2+\pi i x)^2}}$$
$$= \frac{\pi}{2a(1+a^2)\sqrt{1+(\frac{\pi^2}{4}+\ln^2 a)^2}}. \tag{5}$$

For $F(k)=e^{-bk^2}$ one obtains

$$\int_0^{\infty} e^{-bx^2(x^2-\pi^2)}\frac{\cos(2b\pi x^3)\cosh x}{1+2a^2\cosh(2x)+a^4}dx = e^{-b(\pi^2/4+\ln^2 a)^2}\frac{\pi}{4a(1+a^2)}. \tag{6}$$

Another case in which the real part can be made explicit is $F(k)=\cos(\alpha k)$, which leads to

$$\int_0^{\infty} \frac{\cos(\alpha x^2)\cosh(\alpha\pi x)\cosh x}{1+2a^2\cosh(2x)+a^4}dx = \frac{\pi\cos[\alpha(\pi^2/4+\ln^2 a)]}{4a(1+a^2)} \tag{7}$$

as can be verified numerically for $\alpha\pi \leqslant 1$. E.g. for $\alpha=0.1$ and $a=1+2i$, both sides of (7) give $-0.0783703+0.00264214i$. It is interesting that in this case $F(k)$ is the Laplace transform of a distribution. Further variants of (3) and an application to the Riemann Zeta function can be found in [3].

Discussion

All that is required to produce a formula similar to (3) is an integral evaluation where the same integral transform kernel appears under the integral sign on the left and "naked" on the right. For example, (6) and (7) are obvious candidates where the former involves the Laplace transform (wrt b) and the later the cosine transform (wrt α). Neither of these formulas produces anything new. An alternative scheme leading to a similar class of identities has been presented recently[3].

By setting $a = 1$ in (3) one has

$$\int_{-\infty}^{\infty} F(x^2 + i\pi x)\text{sech}x\, dx = \pi F(\pi^2/4). \tag{8}$$

For example,

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh(\pi x)\Gamma(4ax(x+i)+b)} = \frac{1}{\Gamma(a+b)} \tag{9}$$

for real a, b.

A more complicated example is, for $a > 0$, $n \geqslant 1$ and

$$\sigma_{2l} = \sum_{1 \leqslant n_1 < n_2 < \cdots < n_{2l}} \prod_{k \neq n_1, \ldots, n_{2l}} (x^2 + a + \frac{k}{\pi^2})$$

$$\int_{-\infty}^{\infty} \frac{\sum_{l=0}^{\infty}(-1)^l \sigma_{2l} x^{2l} + \left\{ \begin{array}{c} (-1)^{n/2}x^n; n\ even \\ 0; n\ odd \end{array} \right\}}{\prod_{k=0}^{n-1}[(x^2 + a + \frac{k}{\pi^2})^2 + x^2]} \frac{dx}{\cosh(\pi x)}$$
$$= \prod_{k=0}^{n-1} \left(a + \frac{1}{4} + \frac{k}{\pi^2} \right)^{-1}. \tag{10}$$

Anoither expression for the integral (3) is

$$\int_{-\infty}^{\infty} \frac{F[x(x+i)] \cosh(\pi x)}{\cosh[\pi(x+t)] \cosh \pi(x-t)]} dx = \frac{F[t^2 + 1/4)]}{\cosh(\pi t)} \qquad (11).$$

For example, for $a \leqslant b$ and t real

$$\int_{-\infty}^{\infty} \frac{\Gamma[ax(x+i)]}{\Gamma[bx(x+i)]} \frac{\cosh(\pi x)}{\cosh[\pi(x+t)] \cosh[\pi(x-t)]} dx = \frac{\Gamma[a(t^2 + 1/4)]}{\Gamma[b(t^2 + 1/4)]} \text{sech}(\pi t). \tag{12}$$

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Reference

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